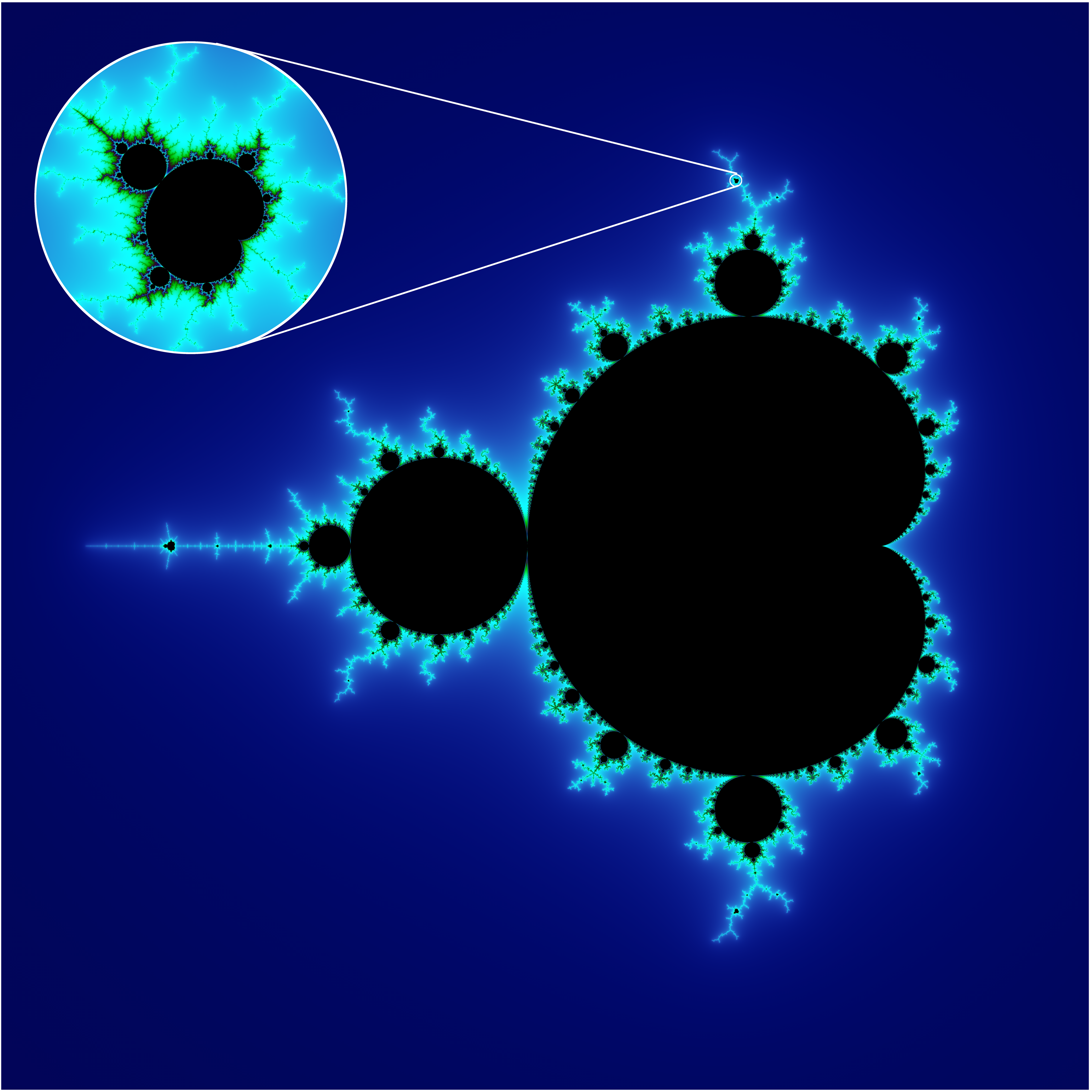


The Mandelbrot Set

For any complex number $c \in \mathbb{C}$, put $p_c(z) = z^2 + c$. Consider the sequence

$$p_c(0) = c, \quad p_c^2(0) = p_c(c) = c^2 + c, \quad p_c^3(c) = c^4 + 2c^3 + c^2 + c, \quad \ldots$$

For some values of c, we have $|p_c^n(0)| \leq 2$ for all n. For all other values of c, it turns out that $p_c^n(0) \to \infty$ as $n \to \infty$. We put

$$M = \{c \in \mathbb{C} \mid |p_c^n(0)| \leq 2 \text{ for all } n \in \mathbb{N}\},$$

and call this the *Mandelbrot set*. It is shown in black in the picture on the left. There is a very rich mathematical theory of the structure of this set. Among the many interesting properties of M, the most obvious is self-similarity: the set contains many scaled-down and slightly distorted copies of itself.

The set M also contains all of its limit points; in other words, it is a closed set. It contains an open set M', which can be defined as follows. We say that z is a *periodic point* for p_c if $p_c^n(z) = z$ for some $n > 0$; the *period* is the smallest n with this property. In this situation, we put $z_k = p_c^k(z)$; these points are again periodic with period n, and only $z_0, \ldots, z_{n-1}$ are distinct, because $z_n = z_0$, $z_{n+1} = z_1$ and so on. We define

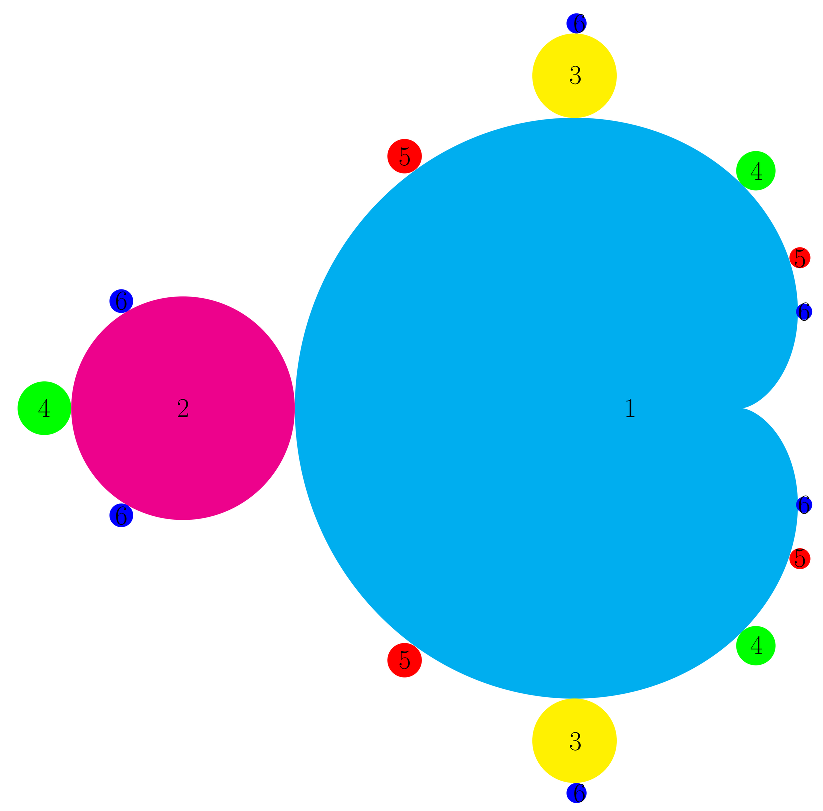
$$\lambda(z) = p_c'(z_0)\, p_c'(z_1) \cdots p_c'(z_{n-1}) = 2^n z_0 z_1 \cdots z_{n-1},$$

and we say that z is *attractive* if $|\lambda(z)| < 1$. We put

$$\begin{aligned} M_n' &= \{c \in C \mid p_c \text{ has an attractive periodic point of period } n\} \\ M' &= M_1' \cup M_2' \cup M_3' \cup M_4' \cup \cdots \\ &= \{c \in C \mid p_c \text{ has an attractive periodic point }\}. \end{aligned}$$

It can be shown that M' is open and contained in M. It is an important conjecture that M' is the *largest* open set contained in M, but this has not yet been proved.
We can define $\mu\colon M' \to \mathbb{C}$ by $\mu(c) = \lambda(z)$, where z is any attractive periodic point for p_c. (A little work is needed to show that this is well-defined.) It can be shown that μ gives a conformal isomorphism from each connected component of M' to the open unit disc D.

Some connected components of M' are shown in the picture below.



The central cardioid, marked 1 in the diagram, is the set M_1'. This is the set of points c such that p_c has an attracting periodic point of period 1, or in other words a point z with $p_c(z) = z$ and $|p_c'(z)| < 1$. Explicitly, we have

$$M_1' = \{\tfrac{1}{2}z - \tfrac{1}{4}z^2 \mid z \in D\}.$$

The map $\mu\colon M_1' \to D$ is just $\mu(\frac{1}{2}z - \frac{1}{4}z^2) = z$.

The set marked 2 is M_2', which consists of points c such that p_c has an attracting periodic point of period 2. It is just a disc of radius $\frac{1}{4}$ with centre at -1, and the map $\mu\colon U_2 \to D$ is just $\mu(-1 + \frac{1}{4}z) = z$.

The two regions marked 3 are components of M_3'. They are approximately circular disks, but not exactly. A formula can be given as follows: for $k = 0, 1, 2$ we define $f_k\colon D \to \mathbb{C}$ by

$$f_k(z) = -\frac{7}{4} - \frac{20}{9}\left(\sinh\left(\frac{2k\pi}{3}i + \frac{1}{3}\operatorname{arcsinh}\left(\frac{88 - 27z}{80\sqrt{5}}\right)\right) - \frac{1}{4\sqrt{5}}\right)^2.$$

The upper region is $f_1(D)$, and the lower region is $f_2(D)$. The region $f_0(D)$ is also part of M_3', but it is not shown in the picture above. It is smaller than the other two regions, and further to the left, and it resembles a cardioid rather than a disc. In all cases we have $\mu(f_k(z)) = z$.

The picture shows three components of M_4', four components of M_5' and six components of M_6'. However, there are many more components for each of these sets. In general, the number of components of M_k' is $c(k) = \sum_{d|k} \mu(k/d) 2^{d-1}$, where μ is the Möbius function:

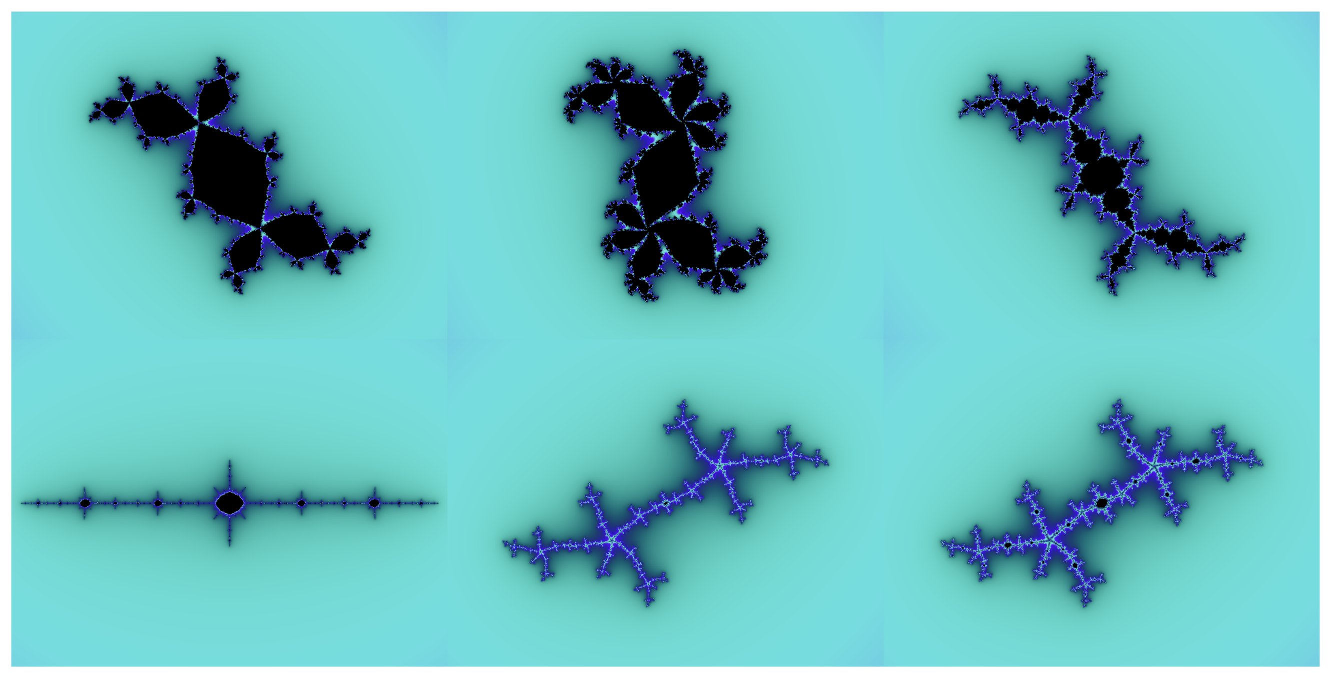
$$\mu(m) = \begin{cases} (-1)^r & \text{if } m \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{if } m \text{ is divisible by the square of any prime.} \end{cases}$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c(k)$	1	1	3	6	15	27	63	120	252	495	1023	2010	4095	8127	16365	32640

For any $c \in M$, we can consider the filled Julia set

$$K_c = \{z \in \mathbb{C} \mid \quad |p_c^n(z)| \leq 4 \text{ for all } n\}.$$

The structure of K_c reflects a great deal of information about the position of c in M and the behaviour of p_c. For example, if p_c has an attractive point of period n, then K_c will have approximate n-fold rotational symmetry. Various Julia sets are shown below.



Poster by Neil Strickland