# Introduction to chromatic homotopy

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## Ordinary cohomology

- For any space $X$ we have a cohomology ring $H^*(X)$
- For many spaces this can be described explicitly: for example, if $X = \{$ two-dimensional subspaces of $\mathbb{C}^4\}$ then $H^*(X) = \mathbb{Z}[c_1, c_2]/(c_1^3 - 2c_1c_2, c_1^2c_2 - c_2^2) = \mathbb{Z}\{c_1^ic_2^j \mid i + j < 3\}$.
- We can also consider the scheme $X_H = \mathrm{spec}(H^*(X))$, so $H^*(X)$ is the ring of functions on $X_H$.
- Now $f : X \to Y$ gives $f_H : X_H \to Y_H$ (depending only on the homotopy class) and $(X \amalg Y)_H = X_H \amalg Y_H$ and $(X \times Y)_H \sim X_H \times Y_H$.
- How good an invariant is this?
  - If $f_H : X_H \to Y_H$ is an isomorphism then $f$ is a homotopy equivalence (subject to mild conditions).
  - The map $[X, Y] \to \mathrm{Schemes}(X_H, Y_H) = \mathrm{Rings}(H^*(Y), H^*(X))$ is typically far from being injective or surjective.
  - If $X_H \simeq Y_H$, that is only weak evidence for $X \simeq Y$.
- How to find better invariants?
  - (a) Use Steenrod operations on $H^*(X; \mathbb{F}_p)$
  - (b) Use generalised cohomology theories.
- But (a) is really part of (b).

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- A *generalised cohomology theory* is a contravariant, homotopy invariant functor $E^*\colon \text{Spaces} \to \text{Rings}^*$ with properties similar to $H^*$, but $E^*(1)$ need not be $\mathbb{Z}$. It takes work to provide interesting examples.
- We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
- Given an even periodic theory $E$ we put $X_E = \text{spf}(E^0 X)$.
- There is an even periodic theory $KU$ with $KU^*(1) = \mathbb{Z}[u, u^{-1}]$ (where $|u| = -2$) and $KU^0(X)$ is the ring of virtual complex vector bundles on $X$.
- Put $MP(n) = \{(v, V) \mid v \in V \leq \mathbb{C}^{2n}\}_\infty$ and $\Sigma^m X = (\mathbb{R}^m \times X)_\infty$ and $MP^k(X) = \varinjlim_n [\Sigma^{2n-k} X, MP(n)]$.

  This gives an even periodic theory with $MP^*(1) = \mathbb{Z}[a_1, a_2, a_3, \dots]$. This is called *periodic complex cobordism*.
- The Nilpotence (pre)Theorem of Hopkins-Devinatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large $k$. This is the most powerful known theorem of the type algebra $\Rightarrow$ topology.
- Fix a prime $p$ and an integer $n > 0$. There is then an even periodic theory $K(p, n)$ with $K(p, n)^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava $K$-theory*.
- The $K(p, n)$'s together carry roughly the same information as $MP$.

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- We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
- Given an even periodic theory $E$ we put $X_E = \text{spf}(E^0X)$.
- There is an even periodic theory $KU$ with $KU^*(1) = \mathbb{Z}[u, u^{-1}]$ (where $|u| = -2$) and $KU^0(X)$ is the ring of virtual complex vector bundles on $X$.
- Put $MP(n) = \{(v, V) \mid v \in V \leq \mathbb{C}^{2n}\}_\infty$ and $\Sigma^m X = (\mathbb{R}^m \times X)_\infty$ and $MP^k(X) = \varinjlim_n [\Sigma^{2n-k}X, MP(n)]$.
  This gives an even periodic theory with $MP^*(1) = \mathbb{Z}[a_1, a_2, a_3, \ldots]$.
  This is called *periodic complex cobordism*.
- The Nilpotence (pre)Theorem of Hopkins-Devinatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large $k$. This is the most powerful known theorem of the type algebra $\Rightarrow$ topology.
- Fix a prime $p$ and an integer $n > 0$. There is then an even periodic theory $K(p, n)$ with $K(p, n)^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava $K$-theory*.
- The $K(p, n)$'s together carry roughly the same information as $MP$.

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# Formal groups — what are they good for?

- Every even periodic theory $E$ gives a formal group $P_E$.
- The functor $E \mapsto P_E$ is not too far from being an equivalence.
- The most elementary examples of formal groups are the additive and multiplicative formal groups; these correspond to $HP$ and $KU$. (Here $HP^i(X) = \prod_j H^{i+2j}(X)$.)
- Steenrod operations in $HP^0(X; \mathbb{F}_p)$ and Adams operations in $KU^0(X)$ are closely related to endomorphisms of the associated formal groups.
- The ring $MP^0(1)$ is naturally isomorphic to the Lazard ring, which plays a central role in formal group theory.
- The Morava $K$-theories $K(p,n)$ all have different formal groups.
- Together with $HP^0(X; \mathbb{F}_p)$ and $HP^0(X; \mathbb{Q})$ this gives all formal groups over fields up to Galois twisting.
- For many spaces $X$ the scheme $X_E$ can be described naturally in terms of $P_E$. For example, if $X = BU(n) = \{n - \text{dimensional subspaces of } \mathbb{C}^\infty\}$ then $X_E = (P_E^n)/\Sigma_n$.

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# Examples of formal groups

- For any ring $R$ we define commutative groups as follows:
  - $G_a(R) = \{a \in R \mid a \text{ is nilpotent }\}$ (under addition)
  - $G_m(R) = \{u \in R \mid u - 1 \text{ is nilpotent }\}$ (under multiplication)
  - $G_r(R) = \{A = \left[\begin{smallmatrix} c & -s \\ s & c \end{smallmatrix}\right] \in M_2(R) \mid c^2 + s^2 = 1,\ c - 1 \text{ nilpotent }\}$
  - $G_e(R) = \{(u, v) \in \text{Nil}(R)^2 \mid v - u^3 + uv^2 = 0\}$ (an elliptic curve)
- These are all functorial in $R$.
- We can define natural bijections $x_i \colon G_i(R) \to \text{Nil}(R)$ by $x_a(a) = a$ and $x_m(u) = u - 1$ and $x_r(A) = s/c$ and $x_e(u, v) = u$ ($v = u^3 - u^7 + O(u^{10})$).
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- The functors $G_i$ are *formal groups*; the power series $F_i$ are *formal group laws*.
- Axioms: $F(s, 0) = s$, $F(s, t) = F(t, s)$ and $F(F(s, t), u) = F(s, F(t, u))$.
- More general version: we have a ground ring $k$, and $G(R)$ is only functorial for $k$-algebras, and $F(s, t) \in k[\![s, t]\!]$.
- Example: for any $a \in k$ we have an FGL $F(s, t) = s + t + ast$ over $k$.

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# The Lazard ring

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- For an FGL we must have $2b_{11}b_{12} + 3b_{13} - 2b_{22}$. In terms of the parameters $a_1 = b_{11}$ and $a_2 = b_{12}$ and $a_3 = b_{22} - b_{13}$ we get $F(s,t) = s+t+a_1st+a_2st(s+t)+2(a_3-a_1a_2)st(s^2+st+t^2)+a_3s^2t^2+O(5)$.
- There are no more relations: any power series of the above form satisfies the FGL conditions up to errors of order 5.
- Lazard's theorem: we can continue to define $a_4, a_5, \dots$ so that $F(s,t)$ can be expressed in terms of the $a_i$, and no further relations are required to make the associativity axiom hold.
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- For an FGL we must have $2b_{11}b_{12} + 3b_{13} - 2b_{22}$. In terms of the parameters $a_1 = b_{11}$ and $a_2 = b_{12}$ and $a_3 = b_{22} - b_{13}$ we get
  $F(s,t) = s+t+a_1st+a_2st(s+t)+2(a_3-a_1a_2)st(s^2+st+t^2)+a_3s^2t^2+O(5)$.
- There are no more relations: any power series of the above form satisfies the FGL conditions up to errors of order 5.
- Lazard's theorem: we can continue to define $a_4, a_5, \dots$ so that $F(s,t)$ can be expressed in terms of the $a_i$, and no further relations are required to make the associativity axiom hold.
- Reformulation: over the Lazard ring $L = \mathbb{Z}[a_1, a_2, \dots]$ there is a universal formal group law $F_u$ such that the resulting map $\text{Rings}(L,k) \to \text{FGL}(k)$ is bijective for all $k$.

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# The chromatic filtration

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