COBORDISM AND FORMAL POWER SERIES

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Thom's cobordism theorem

The graded ring of cobordism classes of manifolds is

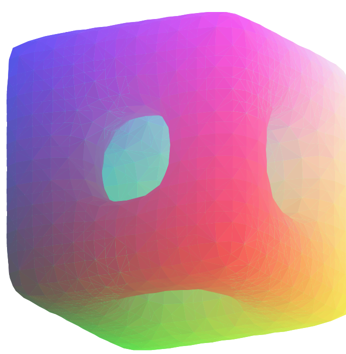
$$\mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{16}, x_{17}, \ldots]$$

with one generator x_k in each degree k not of the form $2^i - 1$.

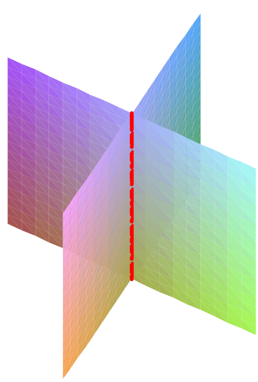
- What does all this mean?
- Why is it interesting?
- How can we prove it?

Manifolds and cobordism

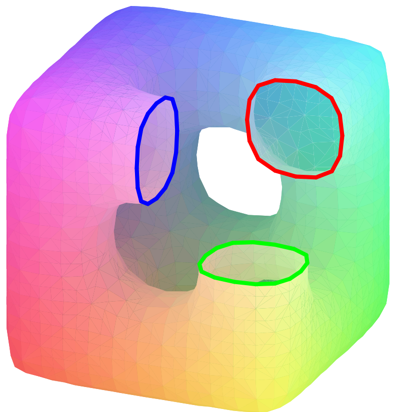
An n-dimensional manifold is a compact space M such that every point has a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$, together with some extra structure to make sense of differentiation.



This surface is not a manifold, because the points on the red line do not have any neighbourhoods homeomorphic to an open ball in $\mathbb{R}^2$.

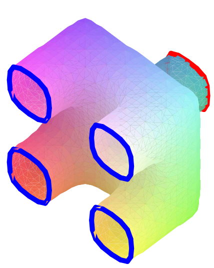


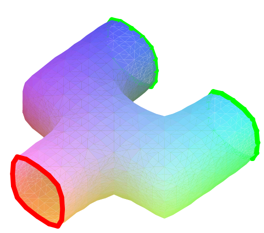
A *manifold with boundary* is like a manifold except that some points are locally like $\mathbb{R}^{n-1} \times [0, \infty)$ rather than $\mathbb{R}^n$.

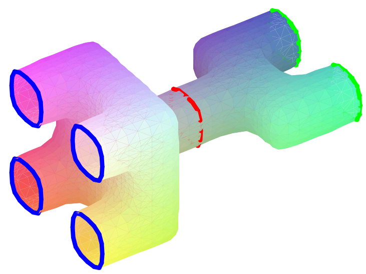


Manifolds M_0 and M_1 are *cobordant* if there is a manifold W with boundary $M_0 \amalg M_1$ (where $\amalg$ means disjoint union).

This is an equivalence relation. In particular, it is associative, as we see by gluing cobordisms:



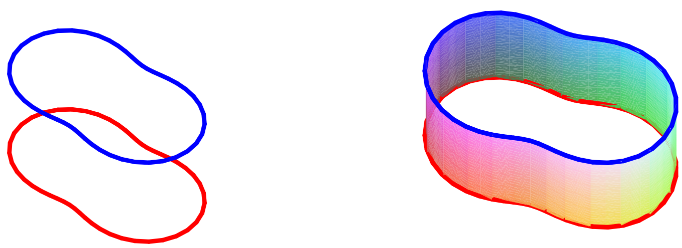




We write MO_n for the set of cobordism classes of n-dimensional manifolds.

We can define addition and multiplication of cobordism classes by $[M_0]+[M_1] = [M_0 \amalg M_1]$ and $[M_0][M_1] = [M_0 \times M_1]$.

Note that $[M] + [M] = 0$ because $M \amalg M$ is the boundary of $M \times [0, 1]$.



This makes MO_* into a graded ring (if $u, v \in MO_n$ and $w \in MO_m$ then $u+v \in MO_n$ and $uv \in MO_{n+m}$).

Why is this interesting?

- Many naturally ocurring spaces are manifolds. It would be nice to classify them in any way we can, but this is very hard.
  - 1-manifolds are just disjoint unions of circles, and are all cobordant to the empty set.
  - 2-manifolds are surfaces, and are classified by their genus and by whether they are orientable. All orientable surfaces are cobordant to the empty set, and all nonorientable surfaces are cobordant to the Klein bottle.
  - There is a rich and elaborate theory of 3-manifolds, which might lead to a complete classification in the future.
  - Very special things happen in dimension 4.
  - In dimensions above 4, there is no hope of a complete classification.

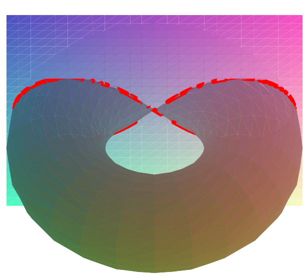
- Spacetime is (perhaps) a manifold, with interesting topology on the very small scale (string theory) and the very large scale (cosmology).

- Given any manifold M, we can pretend that M is spacetime, and see how electromagnetism, or gravitation, or string theory would work out in this context. As this is really a mathematical exercise, we are free to adjust the laws of physics to get better mathematical behaviour if appropriate. If we do this the right way, we can end up with numerical invariants of M that we can use in the classification problem. This has been especially fruitful in dimension 4 (Donaldson theory, Seiberg-Witten theory).
- These "physical" invariants often turn out to depend only on the cobordism class of M. This phenomenon is closely related to Stokes's Theorem: if W has boundary M and ω is a differential form on W then

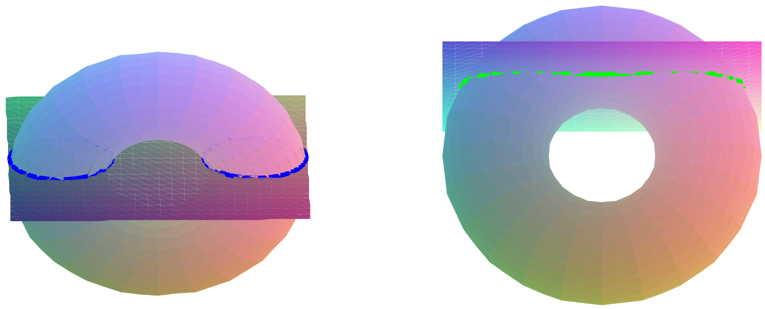
$$\int_W d\omega = \int_{\partial W} \omega = \int_M \omega.$$

- Many interesting geometric constructions give results that are only well-defined up to cobordism, as illustrated below.

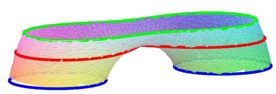
The intersection of two manifolds need not be a manifold.



If we move one or both manifolds a little, then the intersection becomes a manifold. However, the result depends on precisely how we adjust things.



However, the intersection is well-defined up to cobordism:



There are many variations on this theme: constructions that need not give manifolds unless we make an adjustment, where the adjusted result is only well-defined up to cobordism.

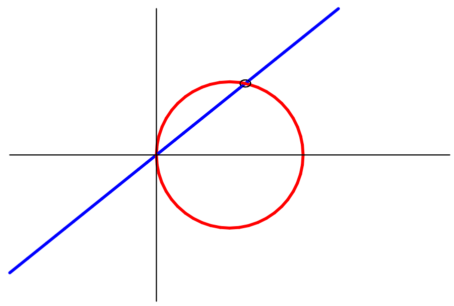
Outline of the proof of Thom's theorem

- Reduction to homotopy theory
  There are spaces $M(m,k)$ such that $MO_n = \pi_{n+k}M(m,k)$ for $m,k \gg 0$.
  More cleanly, there is a spectrum MO such that $MO_n = \pi_n MO$.
  This is a "twisted form" of a certain space BO.
- It is not too hard to calculate the mod 2 homology rings H_*BO, H_*MO and H_*H.
- The algebraic group $\mathrm{spec}(H_*H)$ acts freely on $\mathrm{spec}(H_*MO)$, with quotient $\mathrm{spec}(\mathcal{O}_W)$, where $\mathcal{O}_W$ is the ring in Thom's theorem. All this can be made explicit using formal power series.
- Basic facts about spectra allow us to conclude that $\pi_*MO = \mathcal{O}_W$.

The spaces $B(n,k)$, $E(n,k)$ and $M(n,k)$

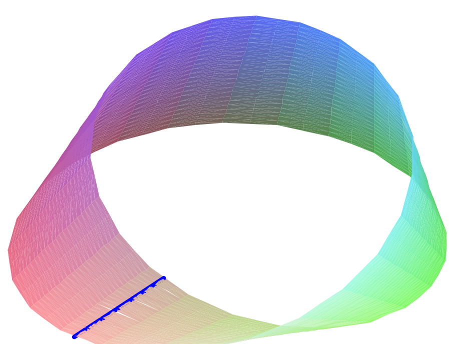
$B(n,k)$ is the space of k-dimensional subspaces of $\mathbb{R}^{n+k}$.

$B(1,1)$ is the space of lines through the origin in $\mathbb{R}^2$. This can be identified with a circle.



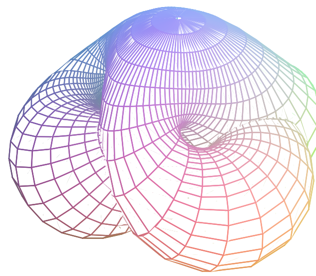
$E(n,k)$ is the space of all pairs (V,v), where V is a k-dimensional subspace of $\mathbb{R}^{n+k}$, and $v \in V$. Note that the subspace where $v=0$ can be identified with a copy of $B(n,k)$.

$E(1,1)$ is the space of pairs consisting of a line in $\mathbb{R}^2$ and a point on that line. It can be identified with a Möbius strip.



The space $M(n,k)$ is obtained by adding a point at infinity to $E(n,k)$, or equivalently, adding a boundary to $E(n,k)$ and collapsing it to a point. Note that $B(n,k) \subset E(n,k) \subset M(n,k)$.

$M(1,1)$ is the Möbius strip with its boundary circle collapsed to a point. This gives the real projective space $\mathbb{R}P^2$.

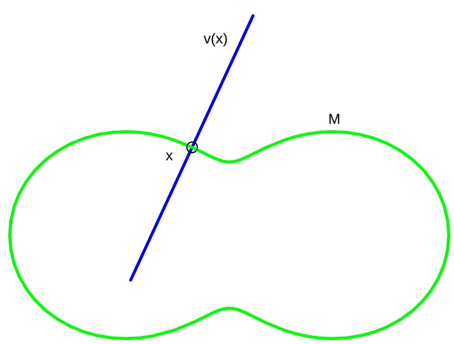


The Pontrjagin-Thom construction

Given a (based) map $f\colon S^{m+k} \to M(n,k)$, the space $f^{-1}(B(n,k))$ will usually be a manifold of dimension m. If it is not, we can make it so by adjusting f slightly. The resulting cobordism class $[M]$ depends only on the homotopy class of f. This construction gives a map

$$\pi_{m+k}(M(n,k)) \to MO_m$$

In the other direction, suppose we start with a manifold M of dimension m. We can harmlessly assume that $M \subset \mathbb{R}^{m+k}$ for some k. For any point $x \in M$, let $\nu(x)$ be the space of vectors in $\mathbb{R}^{m+k}$ that are orthogonal to M at x.



This is a k-dimensional subspace of $\mathbb{R}^{m+k}$, or in other words, a point of the space $B(m,k)$. We thus have a map

$$\nu\colon M \to B(m,k).$$

We also put

$$E(\nu) = \{(x,w) \in M \times \mathbb{R}^{m+k} \mid w \in \nu(x)\}$$

and $T(\nu) = E(\nu) \cup \{\infty\}$. We then have a map

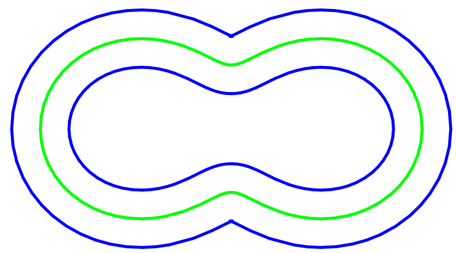
$$\phi\colon T(\nu) \to M(m,k)$$

given by $\phi(x,w) = (\nu(x), w)$ and $\phi(\infty) = \infty$.

Next, we can choose a "tubular neighbourhood" of M in $\mathbb{R}^{m+k}$, or in other words an embedding

$$\theta\colon E(\nu) \to \mathbb{R}^{m+k}$$

such that $\theta(x,w) \approx x+w$ when w is sufficiently small.



This allows us to define a map

$$\theta^{\#}\colon S^{m+k} = \mathbb{R}^{m+k} \cup \{\infty\} \to E(\nu) \cup \{\infty\} = T(\nu)$$

by $\theta^{\#}(\theta(x,w)) = (x,w)$ and $\theta^{\#}(z) = \infty$ for z not in the image of θ. We put this together with our map

$$\phi\colon T(\nu) \to M(m,k)$$

to get a map

$$\phi \circ \theta^{\#}\colon S^{m+k} \to M(m,k).$$

One can show that the homotopy class of this depends only on the cobordism class of the manifold M.

Using the above constructions, one can show that $MO_n = \pi_{n+k}M(m,k)$ when m and k are large relative to n.

We can also introduce the spectrum MO, which is the limit of the spectra $S^{-k} \wedge M(m,k)$ as m and k tend to infinity. This satisfies $MO_n = \pi_n(MO)$. It is closely related to the space BO, which is the limit of the spaces $B(m,k)$ as m and k tend to infinity. We will compute $\pi_*(MO)$ by comparing it with $H_*(MO)$. (All homology groups in this talk have coefficients in $\mathbb{Z}/2$.)

Groups and actions

We now discuss some groups and group actions that will turn out to be relevant for our analysis of $H_*(MO)$ and $\pi_*(MO)$. It turns out to be convenient to discuss these structures as pure algebra first, before making any connection with topology.

Let R be any ring in which $2=0$. We write $R[\![t]\!]$ for the ring of formal power series over R. (A formal power series is any expression of the form $\sum_{i=0}^{\infty} a_i t^i$ with $a_i \in R$. Such series can be manipulated in an obvious way without any consideration of convergence.)

Now put

$$\begin{aligned}
B(R) &= \{f(t) \in R[\![t]\!] \mid f(t) = 1 + O(t)\} \\
T(R) &= \{g(t) \in R[\![t]\!] \mid g(t) = t + O(t^2)\} \\
G(R) &= \{h(t) \in R[\![t]\!] \mid h(s+t) = h(s) + h(t), h(t) = t + O(t^2)\} \\
&= \{\sum_i a_i t^{2^i} \mid a_i \in R,\ a_0 = 1\} \\
W(R) &= \{\sum_j b_j t^{j+1} \mid b_j \in R,\ b_0 = 1,\ b_{2^i-1} = 0 \text{ for } i > 0\} \subseteq T(R)
\end{aligned}$$

Note that $B(R)$ is a group under multiplication and $G(R)$ is a group under composition. Moreover, if $f(t) \in B(R)$ and $h(t) \in G(R)$ then $f(h(t)) \in B(R)$, which shows that $G(R)$ acts on the right on $B(R)$. Similarly, $G(R)$ acts on the right on $T(R)$.

We also see that $B(R)$ acts by multiplication on $T(R)$, and this action is free and transitive.

It can be shown that any $g(t) \in T(R)$ can be written in a unique way as $w(h(t))$, with $w(t) \in W(R)$ and $h(t) \in G(R)$. This shows that $G(R)$ acts freely on $T(R)$, and that the orbit set can be identified with $W(R)$.

Representing rings

Put

$$\begin{aligned} \mathcal{O}_B &= \mathbb{Z}/2[w_1, w_2, w_3, w_4, \dots] \\ \mathcal{O}_T &= \mathbb{Z}/2[x_1, x_2, x_3, x_4, \dots] \\ \mathcal{O}_G &= \mathbb{Z}/2[x_1, x_3, x_7, x_{15}, \dots] \\ \mathcal{O}_W &= \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, x_9, \dots] \end{aligned}$$

(so Thom's theorem is the statement that $MO_* = \mathcal{O}_W$.)

An element of $B(R)$ is a series

$$f(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$$

with $a_i \in R$. Given such a series, we can define a ring map $\alpha_f \colon \mathcal{O}_B \to R$ by $\alpha_f(w_i) = a_i$. This construction gives a bijection between the set $B(R)$ and the set $\mathrm{Hom}(\mathcal{O}_B, R)$ of ring maps from $\mathcal{O}_B$ to R. Similarly, we have

$$\begin{aligned} T(R) &= \mathrm{Hom}(\mathcal{O}_T, R) \\ G(R) &= \mathrm{Hom}(\mathcal{O}_G, R) \\ W(R) &= \mathrm{Hom}(\mathcal{O}_W, R) \end{aligned}$$

It is a standard calculation (one of the most fundamental in algebraic topology) that $H_*(BO) = \mathcal{O}_B$. From this we can deduce (using the Thom isomorphism theorem, which is relatively easy) that $H_*(MO) = \mathcal{O}_T$. There is a spectrum called H with the property that $H_*(X) = \pi_*(H \wedge X)$ for all X. This allows us to define $H_*H = \pi_*(H \wedge H)$. It turns out that $H_*H = \mathcal{O}_G$. This is called the dual Steenrod algebra.

If A is any ring spectrum, then $G(R)$ acts naturally on $\mathrm{Hom}(H_*(A), R)$. If this action is free for all R, one can show that

$$\mathrm{Hom}(\pi_*(A), R) = \mathrm{Hom}(H_*(A), R)/G(R),$$

and that $\pi_*(A)$ is the *unique* ring with this property.

Putting all this together, we see that

$$MO_* = \pi_*(MO) = \mathcal{O}_W = \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, x_9, \dots]$$

This is Thom's cobordism theorem.