

©

Equivariant forms of

Elliptic Cohomology

Problem:

- Given a version  $E$  of elliptic cohomology, construct equivariant versions  $E_A$  for all finite abelian groups  $A$ .
- $E_A$  should relate to  $E$  as  $KU_A$  relates to  $KU$
- If  $E$  is strictly commutative, then  $E_A$  should also be strictly commutative.
- Extension to compact abelian Lie groups should be easy!
- No idea about nonabelian groups.

## Elliptic curves

②

Fix a ring  $k$  & put  $S = \text{spec}(k)$ .

An **elliptic curve** means a commutative group scheme  $C \rightarrow S$  of relative dimension one, with some technical properties.

Locally, non-canonically, after removal of zero:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$\hat{C} :=$  (the part of  $C$  infinitesimally close to zero)

This is a **formal group** over  $S$ .

$$C[n] := \ker(C \xrightarrow{\times n} C)$$

This is a finite flat group scheme, corresponding to a Hopf algebra of rank  $n^2$  as a  $k$ -module.

## Even periodic cohomology

Now suppose we have  $E^*$  for spaces  $Z$  such that

- $E^0(\text{point}) = k$
- $E^1(\text{point}) = 0$
- $\tilde{E}^*(S^{2n}) \cong E^*(\text{point})$  (no shift)

Then  $G = \varinjlim_n \text{spec}(E^0(\mathbb{C}P^n))$

is a formal group over  $S = \text{spec}(k)$ .

- For many spaces  $Z$  there is a simple description of  $E^0 Z$  or  $\text{spec}(E^0 Z)$  in terms of  $G$ .

Eg:  $BU(n)$ ,  $\Omega U(n)$ ,  $B(\mathbb{Z}/d)$ ,  
 $\text{Grass}_d(\mathbb{C}^n)$ , toric varieties, ....



## Elliptic spectra

- An elliptic spectrum is  $(E, C, t)$  where  $E$  is a cohomology theory as before (with  $E^0(\text{point}) = k$ );  $C$  is an elliptic curve over  $S = \text{spec}(k)$  and  $t: G \cong \hat{C}$  (iso. of formal groups).
- There is a forgetful functor from elliptic spectra to elliptic curves. This is not too far from being an equivalence.
- In particular, if  $k$  is a field, a  $\mathbb{Q}$ -algebra, or a ring of the form  $\mathbb{Z}[\frac{1}{n}]$  (with  $n$  even) then any elliptic curve has a canonically associated spectrum.

## Equivariant formal groups (elliptic case)

- Given an elliptic curve  $C$  over  $S$  and a finite abelian group  $A \simeq \bigoplus \mathbb{Z}/d_i$

$$S_A := \text{Hom}(A^*, C) \simeq \prod_i C[d_i]$$

$$C_A = C_S^* \times S_A \quad (\text{an elliptic curve over } S_A)$$

$$\phi_A: A^* \longrightarrow C_A \quad (\text{tautological})$$

$$\bar{C}_A = \left( \begin{array}{l} \text{the part of } C_A \text{ infinitesimally} \\ \text{close to } \phi_A(A^*) \end{array} \right)$$

- $(\bar{C}_A, \phi_A)$  is an example of an equivariant formal group

# Equivariant Formal Groups (general case)

An equivariant formal group over  $S = \text{spec}(k)$  is a formal group scheme  $G = \text{spf}(R)$  equipped with  $\phi: A^* \rightarrow G$  such that

- The zero section in  $G$  is a regular hypersurface (so  $\ker(\varepsilon: R \rightarrow k)$  is free of rank one over  $R$ )
- All of  $G$  is infinitesimally close to the image of  $\phi: A^* \rightarrow G$ .
- A flatness condition

This implies that  $R$  is free of rank  $|A|$  over a subring isomorphic to  $k[[y]]$ , and that  $\Omega_{G/S}^1$  is free of rank one over  $\mathcal{O}_G = R$ .

Example:  $R = \mathbb{Z}[u, x]_{x(x-u)}^\wedge$  ;  $y = x(x-u)$   
 $k = \mathbb{Z}[u]$ .  $G = \text{spf}(R) = \varprojlim_n \text{spec}(R/y^n)$



## Formal groups from cohomology theories

- Put  $\mathcal{U}_A = \mathbb{C}[A]^\infty$  and  $P\mathcal{U}_A = \{1\text{-dim. subspaces}\}$

Then  $\pi_0((P\mathcal{U}_A)^A) = \pi_0\{1\text{-dim subrepresentations}\} = A^*$

giving  $A^* \rightarrow P\mathcal{U}_A$  of  $A$ -spaces

- $P\mathcal{U}_A$  classifies equivariant line bundles  $\therefore$  is a commutative equivariant  $H$ -space.
- Let  $E^*$  be a suitable equivariant cohomology theory then  $(S = \text{spec}(E^0(\text{point})), C_1 = \text{spf}(E^0 P\mathcal{U}_A), \phi = \text{effect of } A^* \rightarrow P\mathcal{U}_A)$  gives an equivariant formal group.
- Conditions:
  - +  $\tilde{E}^* S^V \cong \tilde{E}^* S^0$  (no shift) for any  $\mathbb{C}[A]$ -module  $V$
  - +  $\tilde{E}^0 P\mathcal{U}_A$  free of rank one over  $E^0 P\mathcal{U}_A$ .



## Extension problem

Given a nonequivariant elliptic spectrum  
(elliptic curve  $C$ , cohomology theory  $E$   
with formal group isomorphic to  $\hat{C}$ )

find an equivariant form  $E_A$  such that

$$+ E_A^*(A \times Z) \simeq E^*(Z)$$

+ The associated equivariant formal group  
is  $(S_A, \bar{C}_A, \phi)$  where

$$S_A = \text{Hom}(A^*, C) \quad \text{ie } E_A^0(\text{pt}) = \mathcal{O}_{\text{Hom}(A^*, C)}$$

$$C_A = C \times_S S_A \quad \phi: A^* \rightarrow C_A \text{ tautological}$$

These should have the compatibility property

$$E_A^*(A \times_B Z) = E_B^*(Z)$$

for all  $B$ -spaces  $Z$ .

If  $E$  is strictly commutative, so should  $E_A$  be.

Theorem :

- If  $E$  is  $K(n)$ -local for some  $n$   
(automatically  $n \leq 2$ )  
then  $E_A$  can be constructed for all  $A$ .
- +  $E$  is  $K(0)$ -local iff  $\mathbb{Q} \subseteq k = E^\circ(\text{point})$
- +  $E$  is  $K(1)$ -local if  $p^v = 0$  in  $k$   
for some  $v$  &  $C$  has no supersingular  
fibres. (Not iff, but might as well be.)
- +  $E$  is  $K(2)$ -local if  $p^v = 0$  in  $k$   
for some  $v$  & all fibres of  $C$  are supersingular.
- There is a method to reduce the general  
case to the  $K(n)$ -local case; there  
are some obstructions.

## The rational case

- Put  $k = E^0(\text{point})$ ,  $\omega = \tilde{E}^0(S^2)$

so  $\omega$  is free of rank one over  $k$ .

- If  $\mathbb{Q} \subseteq k$  then  $E^0(Z) \simeq \prod_n H^{2n}(Z; \omega^{\otimes n})$

Also  $\text{Hom}(A^*, C) \simeq \coprod_{B \subseteq A} \text{Mon}(B^*, C)$

Put  $D_A = \mathcal{O}_{\text{Mon}(A^*, C)}$  (a finite free module over  $k$ )

Then  $E_A^0(Z) = \prod_{B \subseteq A} \prod_n D_B \otimes H^{2n}(Z^B; \omega^{\otimes n})$

(this is the unique equivariant form of  $E$  with the required equivariant formal group).

- The definition is inspired by Hopkins-Kuhn-Ravenel, but the precise relationship is indirect.



## Étale group schemes

(11)

- Let  $G$  be a finite flat group scheme over  $S$  (so  $G = \text{spec}(A)$  where  $A$  is a Hopf algebra over  $k$  & a finitely generated projective  $k$ -module).
- $G$  is infinitesimal if the augmentation ideal  $I \subseteq A$  is nilpotent (so all of  $G$  is infinitesimally close to the zero-section).
- $G$  is étale if  $I$  is generated by an idempotent (so  $A = k \times (\text{something})$  as rings so  $G = (\text{zero section}) \amalg (\text{something})$  as schemes so points in  $G$  are definitely zero, or definitely nonzero).
- If  $k$  is a field, then  $\exists$  natural  $\hat{G} \twoheadrightarrow G \twoheadrightarrow Q$  with  $\hat{G}$  infinitesimal,  $Q$  étale.



## Uniformity

- We'll say that an elliptic curve  $C$  is uniform if the group  $C[n] = \ker(C \xrightarrow{n} C)$  fits in  $\hat{C}[n] \rightarrow C[n] \rightarrow Q[n]$  with  $\hat{C}[n]$  infinitesimal &  $Q[n]$  étale (for all  $n$ )
- This is automatic if  $k$  is a field.
- The  $K(m)$ -local cases reduce to the uniform case.
- Put  $Q = \varinjlim_n Q[n]$ , so
 
$$\mathrm{Hom}(A^*, Q) = \mathrm{Hom}(A^*, Q[n]) \text{ for } n \gg 0$$
- Because equality in  $Q$  is unambiguous,
 
$$\mathrm{Hom}(A^*, Q) = \bigsqcup_{B \leq A} \mathrm{Mon}(B^*, Q)$$
- $S_{A,B} :=$  preimage of  $\mathrm{Mon}(B^*, Q)$  in  $S_A = \mathrm{Hom}(A^*, C)$   
 so  $S_A = \bigsqcup_B S_{A,B}$ .

Put  $F_{A,B}^*(Z) = E^*((Z^B)_{h(A/B)})$

Then  $\text{spec}(F_{A,B}^{\circ}(\text{point})) = \text{Hom}((A/B)^*, \hat{C})$

Note that if  $\phi \in S_{A,B}$  then

$$\begin{array}{ccccc} (A/B)^* & \longrightarrow & A^* & \longrightarrow & B^* \\ \phi \downarrow & & \downarrow \phi & & \downarrow \\ \hat{C} & \longrightarrow & C & \longrightarrow & Q \end{array}$$

so  $\text{res} : S_{A,B} \longrightarrow \text{Hom}((A/B)^*, \hat{C}) = \text{spec}(F_{A,B}^{\circ}(\text{pt}))$

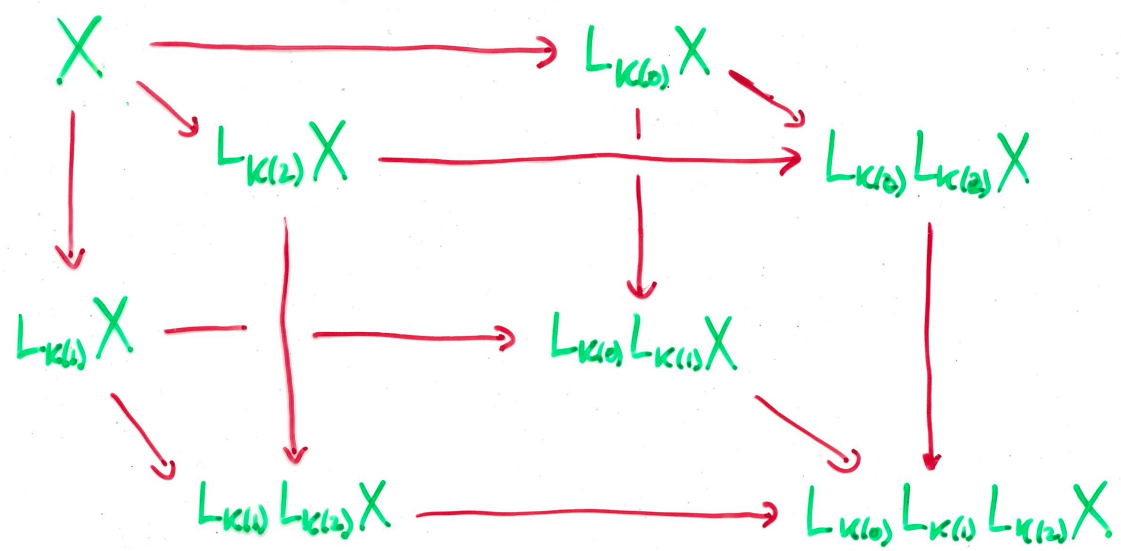
Put  $E_{A,B}^*(Z) = \mathcal{O}_{S_{A,B}} \otimes_{F_{A,B}^{\circ}(\text{pt})} F_{A,B}^*(Z)$

$$E_A^*(Z) = \prod_B E_{A,B}^*(Z)$$

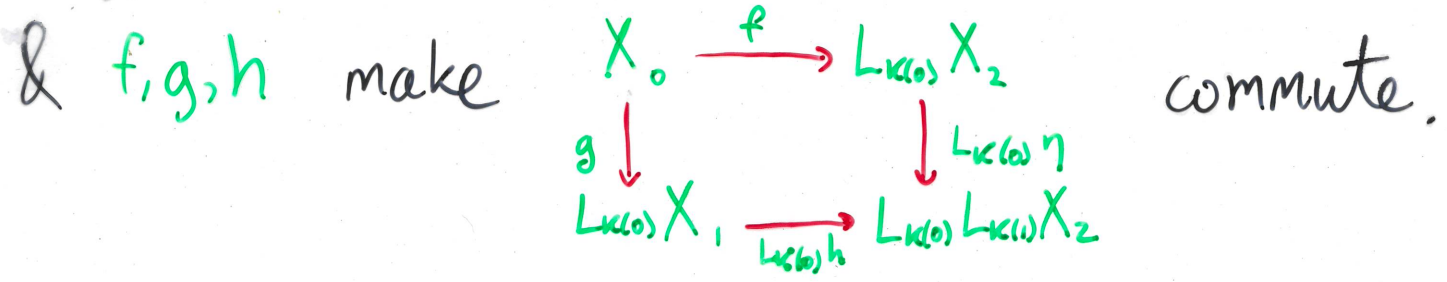
Theorem: this works.

# Chromatic fracture & reconstruction (Hopkins)

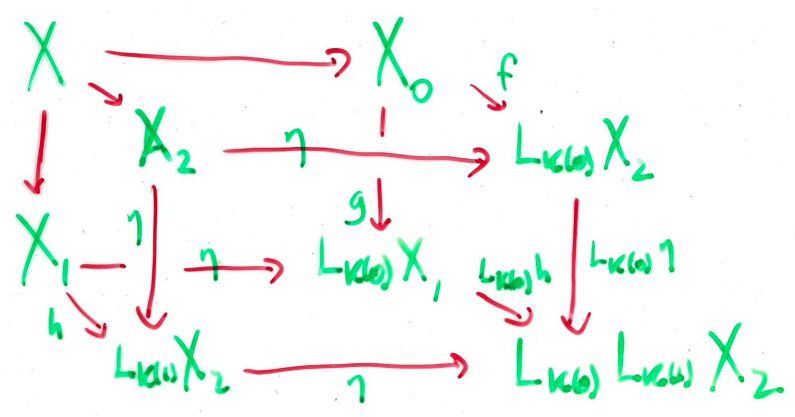
- If  $X$  is  $E(2)$ -local, there is a homotopy - cartesian cube



- Suppose  $X_n$  is  $K(n)$ -local ( $n=0,1,2$ )



Define  $X$  as the homotopy pullback in



Then  $L_{K(n)}X = X_n$  for  $n=0,1,2$ .

Given an elliptic spectrum  $E$ , put

$$X_n = (L_{K(n)} E)_A \quad \text{for } n = 0, 1, 2.$$

One can construct ring maps

$$X_1 \xrightarrow{h} L_{K(1)} X_2, \quad L_{K(0)} X_1 \xleftarrow{g} X_0 \xrightarrow{f} L_{K(0)} X_2$$

making

$$\begin{array}{ccc}
 X_0 & \xrightarrow{\quad} & L_{K(0)} X_2 \\
 \downarrow & & \downarrow \\
 L_{K(0)} X_1 & \xrightarrow{\quad} & L_{K(0)} L_{K(1)} X_2
 \end{array}$$

commute up to homotopy.

Can then adjust to commute on the nose, define  $X$  as a pullback, so

$$L_{K(n)} X = (L_{K(n)} E)_A$$

**BUT** ring structure is tricky.

(In the  $E_\infty$  category, some problems go away, but other ones appear).