Equivariant forms of Elliptic Cohomology

Problem:

- Given a version E of elliptic cohomology, construct equivariant versions E_A for all finite abelian groups A.
- E_A should relate to E as KU_A relates to KU
- If E is strictly commutative, then E_A should also be strictly commutative.
- Extension to compact abelian Lie groups should be easy!
- No idea about nonabelian groups.

Elliptic curves

Fix a ring k & put $S = \mathrm{spec}(k)$.

An elliptic curve means a commutative group scheme $C \to S$ of relative dimension one, with some technical properties.

Locally, non-canonically, after removal of zero:

$$y^2 + a_1 x y + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$$\hat{C} := \left(\begin{array}{l} \text{the part of } C \text{ infinitesimally} \\ \text{close to zero} \end{array} \right)$$

This is a formal group over S.

$$C[n] := \ker(C \xrightarrow{\times n} C)$$

This is a finite flat group scheme, corresponding to a Hopf algebra of rank n^2 as a k-module.

Even periodic cohomology

Now suppose we have E^*Z for spaces Z such that

- $E^0(\text{point}) = k$
- $E^1(\text{point}) = 0$
- $\tilde{E}^*(S^{2n}) \simeq E^*(\text{point})$ (no shift)

Then $G = \varinjlim_n \operatorname{spec}(E^0\mathbb{C}P^n)$ is a formal group over $S = \operatorname{spec}(k)$.

- For many spaces Z there is a simple description of E^0Z or $\operatorname{spec}(E^0Z)$ in terms of G.

  Eg: $BU(n)$, $\Omega U(n)$, $B(\mathbb{Z}/d)$, $\mathrm{Grass}_d(\mathbb{C}^n)$, toric varieties, ...

Elliptic spectra

- An elliptic spectrum is (E, C, t) where E is a cohomology theory as before (with $E^0(\text{point}) = k$); C is an elliptic curve over $S = \operatorname{spec}(k)$ and $t : G \simeq \hat{C}$ (iso. of formal groups).
- There is a forgetful functor from elliptic spectra to elliptic curves.

This is not too far from being an equivalence.

- In particular, if k is a field, a $\mathbb{Q}$-algebra, or a ring of the form $\mathbb{Z}[\frac{1}{n}]$ (with n even) then any elliptic curve has a canonically associated spectrum.

Equivariant formal groups (elliptic case)

- Given an elliptic curve C over S and a finite abelian group $A \simeq \bigoplus \mathbb{Z}/d_i$

$$S_A := \mathrm{Hom}(A^*, C) \simeq \prod_i C[d_i]$$

$$C_A = C \times_S S_A \quad \text{(an elliptic curve over } S_A\text{)}$$

$$\phi_A : A^* \longrightarrow C_A \quad \text{(tautological)}$$

$$\overline{C}_A = \left(\begin{array}{l}\text{the part of } C_A \text{ infinitesimally} \\ \text{close to } \phi_A(A^*)\end{array}\right).$$

- $(\overline{C}_A, \phi_A)$ is an example of an **equivariant formal group**

Equivariant Formal Groups (general case)

An *equivariant formal group* over $S = \mathrm{spec}(k)$ is a formal group scheme $G = \mathrm{spf}(R)$ equipped with $\phi : A^* \to G$ such that

- The zero section in G is a regular hypersurface (so $\ker(\varepsilon : R \to k)$ is free of rank one over R)
- All of G is infinitesimally close to the image of $\phi : A^* \to G$.
- A flatness condition

This implies that R is free of rank $|A|$ over a subring isomorphic to $k[\![y]\!]$, and that $\Omega^1_{G/S}$ is free of rank one over $\mathcal{O}_G = R$.

Example: $R = \mathbb{Z}[u,x]^{\wedge}_{x(x-u)}$; $y = x(x-u)$

$k = \mathbb{Z}[u]$. $\quad G = \mathrm{spf}(R) = \varinjlim_n \mathrm{spec}(R/y^n)$

Formal groups from cohomology theories

- Put $\mathcal{U}_A = \mathbb{C}[A]^\infty$ and $P\mathcal{U}_A = \{\text{1-dim. subspaces}\}$

Then $\pi_0((P\mathcal{U}_A)^A) = \pi_0\{\text{1-dim subrepresentations}\} = A^*$

giving $A^* \longrightarrow P\mathcal{U}_A$ of A-spaces

- $P\mathcal{U}_A$ classifies equivariant line bundles $\therefore$ is a commutative equivariant H-space.

- Let E^* be a suitable equivariant cohomology theory then $(S = \mathrm{spec}(E^0(\text{point})),\ G = \mathrm{spf}(E^0 P\mathcal{U}_A),$

$$\phi = \text{effect of } A^* \to P\mathcal{U}_A)$$

gives an equivariant formal group.

- Conditions:
  - + $\tilde{E}^* S^V \simeq \tilde{E}^* S^0$ (no shift) for any $\mathbb{C}[A]$-module V
  - + $\tilde{E}^0 P\mathcal{U}_A$ free of rank one over $E^0 P\mathcal{U}_A$.

Extension problem

- Given a nonequivariant elliptic spectrum (elliptic curve C, cohomology theory E with formal group isomorphic to $\hat{C}$) find an equivariant form E_A such that
  + $E_A^*(A \times Z) \simeq E^*(Z)$
  + The associated equivariant formal group is $(S_A, \overline{C}_A, \phi)$ where

$$S_A = \mathrm{Hom}(A^*, C) \quad \text{ie} \quad E_A^0(pt) = \mathcal{O}_{\mathrm{Hom}(A^*, C)}.$$

$$C_A = C \times_S S_A \qquad \phi : A^* \to C_A \text{ tautological}$$

- These should have the compatibility property

$$E_A^*(A \times_B Z) = E_B^*(Z)$$

for all B-spaces Z.
- If E is strictly commutative, so should E_A be.

Theorem :

- If E is $K(n)$-local for some n (automatically $n \leq 2$) then E_A can be constructed for all A.
- + E is $K(0)$-local iff $\mathbb{Q} \subseteq k = E^0(\text{point})$
  + E is $K(1)$-local if $p^v = 0$ in k for some v & C has no supersingular fibres. (Not iff, but might as well be)
  + E is $K(2)$-local if $p^v = 0$ in k for some v & all fibres of C are supersingular.
- There is a method to reduce the general case to the $K(n)$-local case; there are some obstructions.

The rational case

- Put $k = E^0(\text{point})$, $\omega = \tilde{E}^0(S^2)$

  so ω is free of rank one over k.

- If $\mathbb{Q} \subseteq k$ then $E^0(Z) \simeq \prod_n H^{2n}(Z; \omega^{\otimes n})$

  Also $\mathrm{Hom}(A^*, C) \simeq \coprod_{B \leq A} \mathrm{Mon}(B^*, C)$

  Put $D_A = \mathcal{O}_{\mathrm{Mon}(A^*, C)}$ (a finite free module over k)

  Then $E_A^0(Z) = \prod_{B \leq A} \prod_n D_B \otimes H^{2n}(Z^B; \omega^{\otimes n})$

  (this is the unique equivariant form of E with the required equivariant formal group).

- The definition is inspired by Hopkins-Kuhn-Ravenel, but the precise relationship is indirect.

Etale group schemes

- Let G be a finite flat group scheme over S (so $G = \mathrm{spec}(A)$ where A is a Hopf algebra over k & a finitely generated projective k-module).

- G is infinitesimal if the augmentation ideal $I \leq A$ is nilpotent (so all of G is infinitesimally close to the zero-section).

- G is étale if I is generated by an idempotent (so $A = k \times (\text{something})$ as rings so $G = (\text{zero section}) \amalg (\text{something})$ as schemes so points in G are definitely zero, or definitely nonzero).

- If k is a field, then $\exists$ natural $\hat{G} \rightarrowtail G \twoheadrightarrow Q$ with $\hat{G}$ infinitesimal, Q étale.

Uniformity

- We'll say that an elliptic curve C is **uniform** if the group $C[n] = \ker(C \xrightarrow{n} C)$ fits in $\hat{C}[n] \rightarrowtail C[n] \twoheadrightarrow Q[n]$ with $\hat{C}[n]$ infinitesimal & $Q[n]$ étale (for all n)

- This is automatic if k is a field.

- The $K(m)$-local cases reduce to the uniform case.

- Put $Q = \varinjlim_n Q[n]$, so

$$\mathrm{Hom}(A^*, Q) = \mathrm{Hom}(A^*, Q[n]) \quad \text{for } n \gg 0$$

- Because equality in Q is unambiguous,

$$\mathrm{Hom}(A^*, Q) = \coprod_{B \leq A} \mathrm{Mon}(B^*, Q)$$

- $S_{A,B} :=$ preimage of $\mathrm{Mon}(B^*, Q)$ in $S_A = \mathrm{Hom}(A^*, C)$

so $S_A = \coprod_B S_{A,B}$.

Put $F^*_{A,B}(Z) = E^*((Z^B)_{h(A/B)})$

Then $\operatorname{spec}(F^0_{A,B}(\text{point})) = \operatorname{Hom}((A/B)^*, \hat{C})$

Note that if $\phi \in S_{A,B}$ then

$$\begin{tikzcd}
(A/B)^* \arrow[r, tail] \arrow[d, "\phi"'] & A^* \arrow[r, two heads] \arrow[d, "\phi"] & B^* \arrow[d] \\
\hat{C} \arrow[r] & C \arrow[r] & Q
\end{tikzcd}$$

so $\operatorname{res} : S_{A,B} \longrightarrow \operatorname{Hom}((A/B)^*, \hat{C}) = \operatorname{spec}(F^0_{A,B}(pt))$.

Put $E^*_{A,B}(Z) = \mathcal{O}_{S_{A,B}} \otimes_{F^0_{A,B}(pt)} F^*_{A,B}(Z)$

$$E^*_A(Z) = \prod_B E^*_{A,B}(Z)$$

Theorem: this works.

Chromatic fracture & reconstruction (Hopkins)

- If X is $E(2)$-local, there is a homotopy-cartesian cube

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & L_{K(0)}X \arrow[dr] \arrow[dd] & \\
& L_{K(2)}X \arrow[rr] \arrow[dd] & & L_{K(0)}L_{K(2)}X \arrow[dd] \\
L_{K(1)}X \arrow[rr] \arrow[dr] & & L_{K(0)}L_{K(1)}X \arrow[dr] & \\
& L_{K(1)}L_{K(2)}X \arrow[rr] & & L_{K(0)}L_{K(1)}L_{K(2)}X
\end{tikzcd}
$$

- Suppose X_n is $K(n)$-local $(n = 0,1,2)$ & f, g, h make

$$
\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "g"'] & L_{K(0)}X_2 \arrow[d, "L_{K(0)}\eta"] \\
L_{K(0)}X_1 \arrow[r, "L_{K(0)}h"'] & L_{K(0)}L_{K(1)}X_2
\end{tikzcd}
$$

commute.

Define X as the homotopy pullback in

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & X_0 \arrow[dr, "f"] \arrow[dd, "g"] & \\
& X_2 \arrow[rr, "\eta"] \arrow[dd, "\eta"] & & L_{K(0)}X_2 \arrow[dd, "L_{K(0)}\eta"] \\
X_1 \arrow[rr, "\eta"] \arrow[dr, "h"'] & & L_{K(0)}X_1 \arrow[dr, "L_{K(0)}h"] & \\
& L_{K(1)}X_2 \arrow[rr, "\eta"'] & & L_{K(0)}L_{K(1)}X_2
\end{tikzcd}
$$

Then $L_{K(n)}X = X_n$ for $n = 0,1,2$.

Given an elliptic spectrum E, put

$X_n = (L_{K(n)} E)_A$ for $n = 0, 1, 2$.

One can construct ring maps

$X_1 \xrightarrow{h} L_{K(1)} X_2$, $L_{K(0)} X_1 \xleftarrow{g} X_0 \xrightarrow{f} L_{K(0)} X_2$

making
$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & L_{K(0)} X_2 \arrow[d] \\ L_{K(0)} X_1 \arrow[r] & L_{K(0)} L_{K(1)} X_2 \end{tikzcd}$$
commute up to homotopy.

Can then adjust to commute on the nose, define X as a pullback, so

$$L_{K(n)} X = (L_{K(n)} E)_A$$

BUT ring structure is tricky.

(In the E_∞ category, some problems go away, but other ones appear).