An example in the geometry of surfaces

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July 14, 2016

Surfaces in S^3 have a rich structure

Let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be any smooth surface of genus $g > 1$.

- X separates S^3 into two handlebodies, which are homeomorphic to each other and homotopy equivalent to a wedge of g circles.
- X inherits an orientation and a metric from S^3.
- We can define $J_x \colon T_xX \to T_xX$ to be a $\frac{1}{4}$ turn anticlockwise; this satisfies $J_x^2 = -1$ and so makes T_xX a complex vector space of dimension one.
- X can be covered by open sets U for which there is a diffeomorphism $f \colon U \to D = \{z \in \mathbb{C} \mid |z| < 1\}$ whose derivative is $\mathbb{C}$-linear. This makes X a one-dimensional complex manifold, or in other words a Riemann surface.
- Any compact Riemann surface is isomorphic to a projective algebraic curve, or a branched cover of the Riemann sphere.
- Any compact connected Riemann surface can be constructed from a polygon by identifying edges in pairs.
- Any compact connected Riemann surface of genus $g > 1$ is the quotient of the unit disc by the discrete action of a Fuchsian group.

For many of these phenomena, the literature contains no explicit examples.

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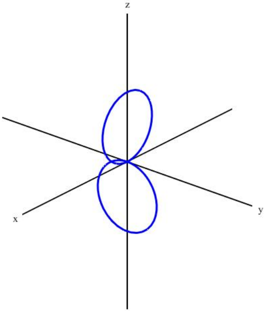

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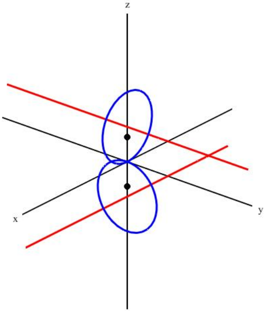

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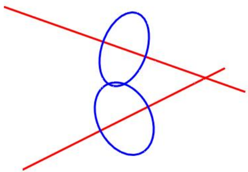

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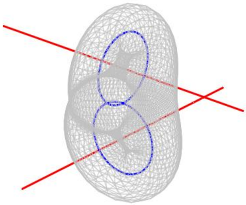

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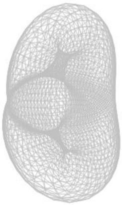

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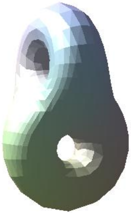

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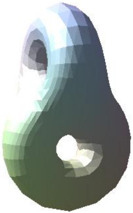

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We define a group G as follows:

$$G = \langle \lambda, \mu, \nu \mid \lambda^4 = \mu^2 = \nu^2 = (\mu\nu)^2 = (\lambda\mu)^2 = (\lambda\nu)^2 = 1 \rangle$$
$$= \{\lambda^i \mu^j \nu^k \mid 0 \leq i < 4,\ 0 \leq j, k < 2\}$$

We write V^* for $\{0, \ldots, 13\}$ with G acting by

$$\lambda \mapsto (2\ 3\ 4\ 5)\ (6\ 7\ 8\ 9)\ (10\ 11)\ (12\ 13)$$
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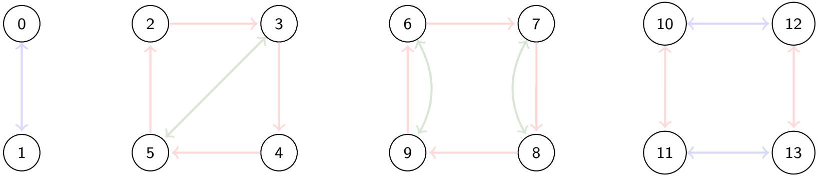

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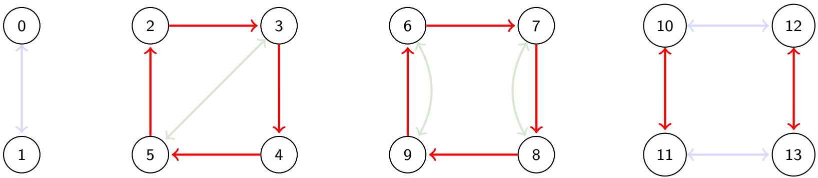

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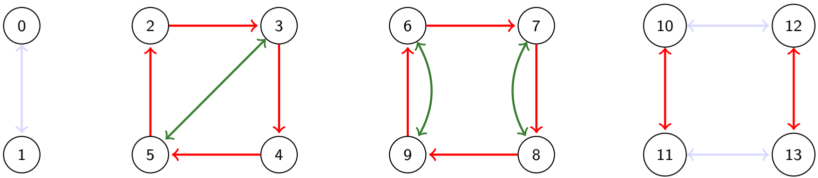

Action of ν

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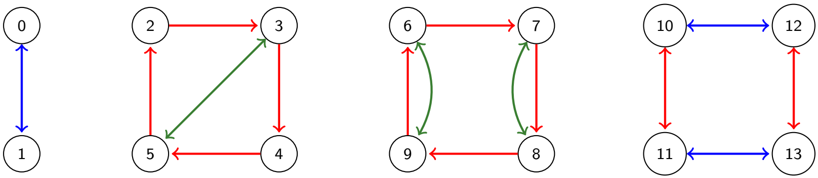

Action of $\mu\nu$

Definition: A *precromulent surface* is a compact Riemann surface X of genus two with an action of G such that

(a) The elements λ and μ act conformally, and the element of ν acts anticonformally.

(b) The set $V = \{v \in X \mid \mathrm{stab}_{\langle \lambda, \mu \rangle}(v) \neq 1\}$ is isomorphic to V^* as a G-set.

A *precromulent labelling* of X is a specific choice of isomorphism $V^* \simeq V$, or equivalently, a listing of the points in V as $v_0, \dots, v_{13}$ such that G permutes these points in accordance with the permutations listed on the last slide.
A *cromulent labelling* is a precromulent labelling such that

(c) λ acts on the tangent space $T_{v_0}X$ as multiplication by i.

(d) In the set $X' = \{x \in X \mid \mathrm{stab}_G(x) = 1\}$, there is a connected component F' whose closure contains $\{v_0, v_3, v_6, v_{11}\}$.

One can show that every precromulent surface has precisely two cromulent labellings, which are exchanged by the action of λ^2. A *cromulent surface* is a precromulent surface with a choice of cromulent labelling.

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Adjective

cromulent (*not comparable*)

1. Fine, acceptable or normal; excellent, realistic, legitimate or authentic. [quotations ▲]
   - **1996** February 18, Matt Groening et al., "Lisa the Iconoclast", *The Simpsons* season 7 episode 16:

     Mrs. Krabappel: Embiggens? I never heard that word before moving to Springfield.
     Ms. Hoover: I don't know why, it's a perfectly **cromulent** word.

     [...]

     Principal Skinner: He's embiggened that role with his **cromulent** performance.

The embedded family

For $a \in (0,1)$, put

$$EX(a) = \{x \in S^3 \mid ((a^{-2}+1)x_3^2 - 2)x_4 + a^{-1}(x_1^2 - x_2^2)x_3 = 0\}.$$

$$\begin{aligned}
\lambda(x_1,x_2,x_3,x_4) &= (-x_2, \quad x_1, \quad x_3, -x_4) \\
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v_0 &= (\ \ 0,\ \ 0,\ \ 1,\ \ 0) & v_6 &= (\ \ 1,\ \ 1,\ \ 0,\ \ 0)/\sqrt{2} \\
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v_4 &= (-1,\ \ 0,\ \ 0,\ \ 0) & v_{10} &= (0,0,\ \ \sqrt{2}a,\ \ \sqrt{1-a^2})/\sqrt{1+a^2} \\
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Special features for $a = 1/\sqrt{2}$

- The complexification $CEX(a)$ is smooth for $a \neq 1/\sqrt{2}$, but when $a = 1/\sqrt{2}$ it is isomorphic to Cayley's singular cubic:

$$X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4 = 0$$

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The projective family

For $a \in (0,1)$ put

$$PX_0(a) = \{(w,z) \in \mathbb{C}^2 \mid w^2 = z^5 - (a^2 + a^{-2})z^3 + z\}.$$

Normalization adds a point at ∞ to give a smooth projective curve $PX(a)$. Let G act by

$$\lambda(w,z) = (iw,\ -z) \qquad \mu(w,z) = (-w/z^3,\ 1/z) \qquad \nu(w,z) = (\overline{w},\ \overline{z}).$$

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v_5 = (\ \ i(a^{-1} - a),\ \ 1) & v_9 = \left(\ \ \overline{\omega}(a^{-1} + a), -i\right) & v_{13} = (0,\ \ a^{-1}).
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(where $\omega = e^{i\pi/4}$). Then $PX(a)$ is cromulent.

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(where $\omega = e^{i\pi/4}$). Then $PX(a)$ is cromulent.

The projective family

For $a \in (0,1)$ put

$$PX_0(a) = \{(w,z) \in \mathbb{C}^2 \mid w^2 = z^5 - (a^2 + a^{-2})z^3 + z\}.$$

Normalization adds a point at ∞ to give a smooth projective curve $PX(a)$.
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$$\lambda(w,z) = (iw,\ -z) \qquad \mu(w,z) = (-w/z^3,\ 1/z) \qquad \nu(w,z) = (\overline{w},\ \overline{z}).$$

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v_0 = (0,0) & v_1 = \infty & \\
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The hyperbolic family

Define a group Π as follows:

$$\Pi = \langle \beta_i \mid i \in \mathbb{Z}/8 \rangle / \langle \beta_i \beta_{i+4}, \beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 \rangle$$

Given $a \in (0,1)$ put $a_\pm = \sqrt{1 \pm a^2}$,
and define automorphisms of $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ by

$$\lambda(z) = iz \qquad \beta_0(z) = \frac{a_+ z + 1}{z + a_+}$$

$$\mu(z) = \frac{a_+ z - a^2 - i}{(a^2 - i)z - a_+} \qquad \beta_1(z) = \frac{a_+^3 z - (2+i)a^2 - i}{((i-2)a^2 + i)z + a_+^3}$$

$$\nu(z) = \bar{z} \qquad \beta_{2n}(z) = i^n \beta_0(z/i^n)$$

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These give an action of Π on Δ, and an action of G on $HX(a) = \Delta/\Pi$.
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Maple code

- It is strenuous and error-prone to verify the cromulence axioms for $HX(a)$ by hand.
- Some other verifications, to be discussed later, are even more strenuous.
- We have instead used Maple. The project has 30000 lines of Maple code, some for numerical calculation and visualization, some for symbolic verification. There is a systematic framework which checks thousands of assertions.
- (By comparison, the 165 page memoir describing the project is generated by 15000 lines of LaTeX.)
- This does not quite reach the same level of rigour as proof assistants like Agda or Isabelle, but it is a major step in that direction.
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Universality

Theorem: For any cromulent X, there is a unique a_P such that there is a (unique) cromulent isomorphism $X \to PX(a_P)$.

Proof: An isotropy calculation shows that $X/\langle\lambda^2\rangle$ has genus 0, and so is isomorphic to $\mathbb{C}_\infty$; one can arrange that $v_0 \mapsto 0$ and $v_1 \mapsto \infty$ and $v_3 \mapsto 1$; then the image of v_{10} determines a_P. □

Theorem: For any cromulent X, there is a unique a_H such that there is a (unique) cromulent isomorphism $HX(a_H) \to X$.

(Here the proof is quite intricate, but the ingredients are fairly standard.)

Conjecture: The embedded family is also universal in the same sense.

Theorem: We have $EX^* \simeq HX(a_H) \simeq PX(a_P)$, where $a_H \simeq 0.8005319$ and $a_P \simeq 0.0983562$.

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Anticonformal involutions

It is a general fact that if X is a compact Riemann surface, and $\alpha\colon X \to X$ is an anticonformal involution, then the fixed set X^α is a finite disjoint union of smoothly embedded circles.
Thus, in a cromulent surface X, these sets are circles:

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\begin{aligned}
C_0 &= \text{ the component of } v_2 \text{ in } X^{\mu\nu} \\
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Curve systems

By a *curve system* on a cromulent surface X, we mean a family of real analytic embeddings $c_k \colon \mathbb{R}/2\pi\mathbb{Z} \to X$ (for $0 \leq k \leq 8$) with values

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0			0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{4}$				
1	0	π					$\frac{\pi}{2}$		$-\frac{\pi}{2}$					
2	0	π						$\frac{\pi}{2}$		$-\frac{\pi}{2}$				
3				$\frac{\pi}{2}$		$-\frac{\pi}{2}$						0		π
4			$-\frac{\pi}{2}$		$\frac{\pi}{2}$						0		π	
5	0											π		
6	0										π			
7		0												π
8		0											π	

and equivariance

$$
\begin{array}{lll}
\lambda(c_0(t)) = c_0(t + \pi/2) & \mu(c_0(t)) = c_0(-t) & \nu(c_0(t)) = c_0(-t) \\
\lambda(c_1(t)) = c_2(t) & \mu(c_1(t)) = c_2(t + \pi) & \nu(c_1(t)) = c_2(-t) \\
\lambda(c_2(t)) = c_1(-t) & \mu(c_2(t)) = c_1(t + \pi) & \nu(c_2(t)) = c_1(-t) \\
\lambda(c_3(t)) = c_4(t) & \mu(c_3(t)) = c_3(t + \pi) & \nu(c_3(t)) = c_3(-t) \\
\lambda(c_4(t)) = c_3(-t) & \mu(c_4(t)) = c_4(-t - \pi) & \nu(c_4(t)) = c_4(t) \\
\lambda(c_5(t)) = c_6(t) & \mu(c_5(t)) = c_7(t) & \nu(c_5(t)) = c_5(t) \\
\lambda(c_6(t)) = c_5(-t) & \mu(c_6(t)) = c_8(-t) & \nu(c_6(t)) = c_6(-t) \\
\lambda(c_7(t)) = c_8(t) & \mu(c_7(t)) = c_5(t) & \nu(c_7(t)) = c_7(t) \\
\lambda(c_8(t)) = c_7(-t) & \mu(c_8(t)) = c_6(-t) & \nu(c_8(t)) = c_8(-t)
\end{array}
$$

Every cromulent surface admits a curve system, and $\text{image}(c_k) = C_k$.

A curve system for EX^*

We can define $c_0, \ldots, c_8 \colon \mathbb{R}/2\pi\mathbb{Z} \to EX^*$ as follows:

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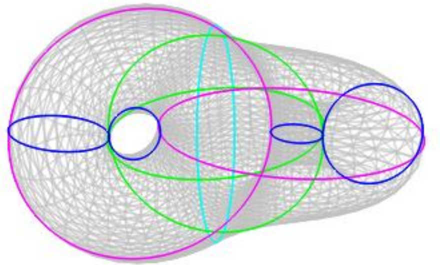

c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

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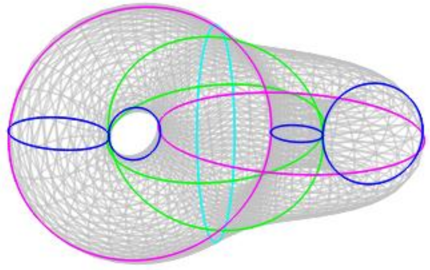

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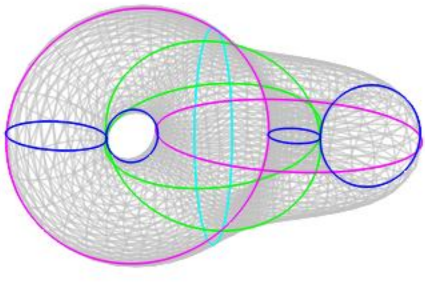

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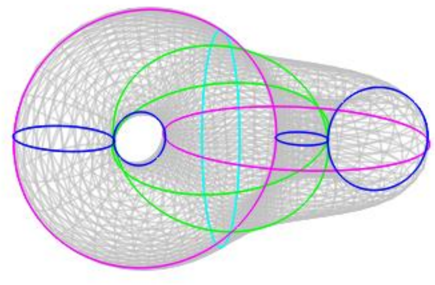

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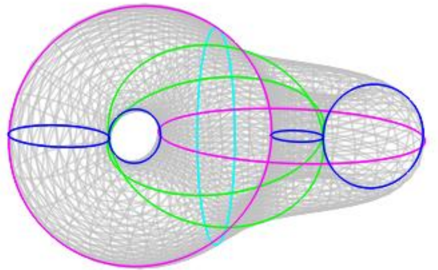

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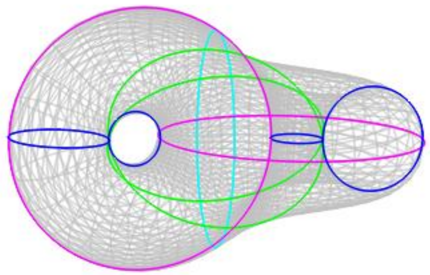

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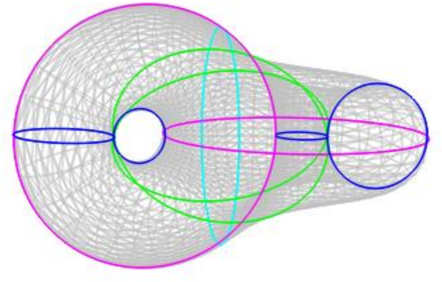

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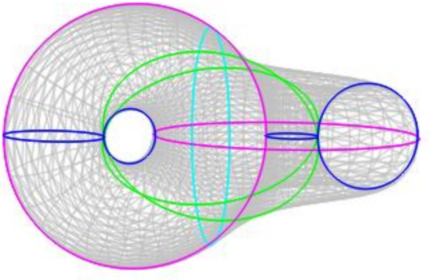

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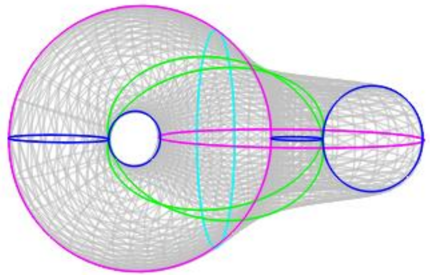

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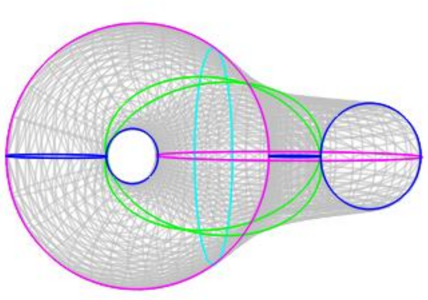

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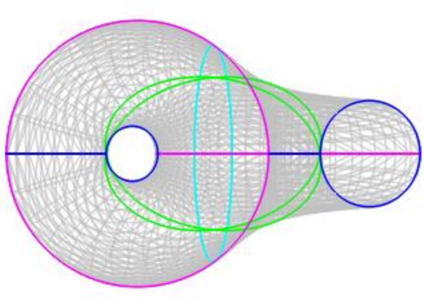

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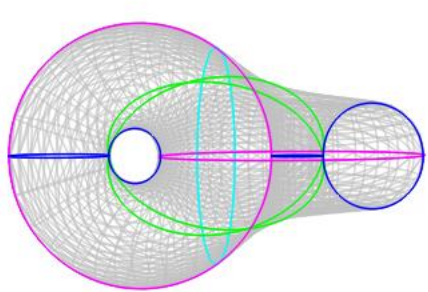

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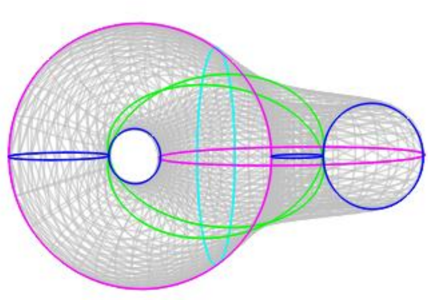

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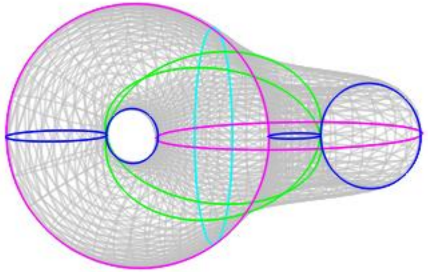

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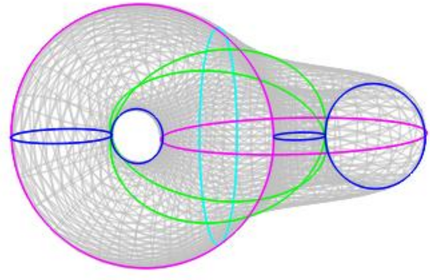

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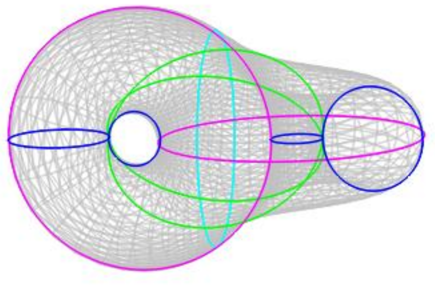

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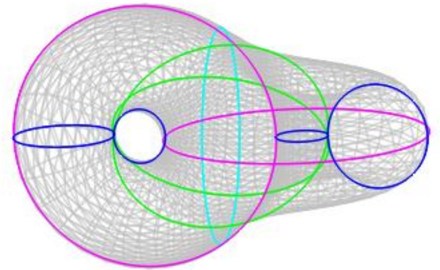

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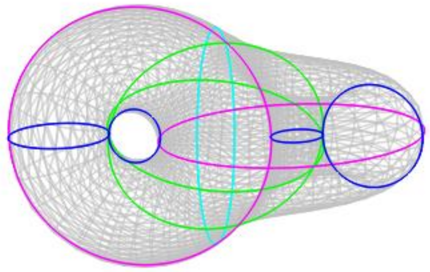

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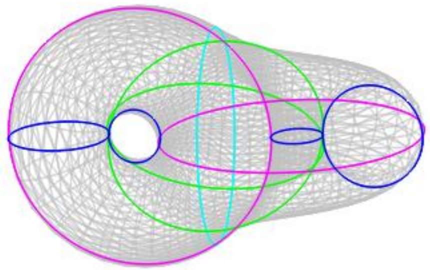

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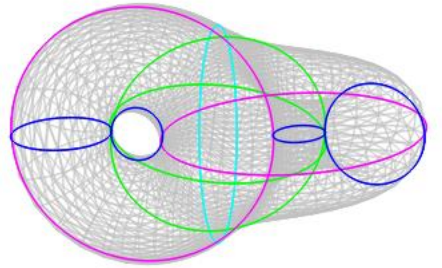

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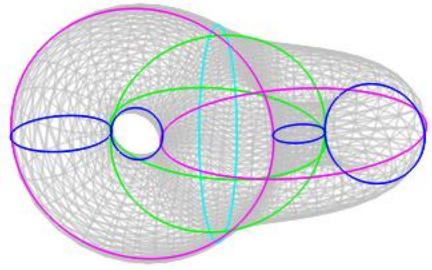

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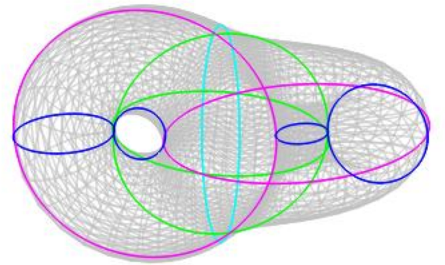
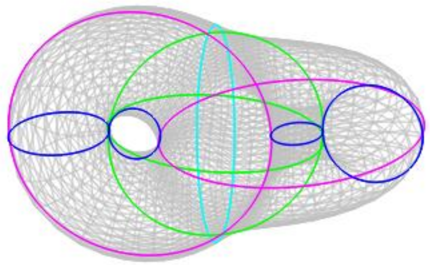


c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



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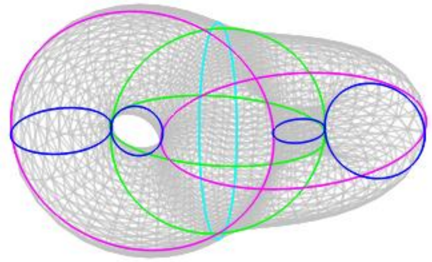

c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

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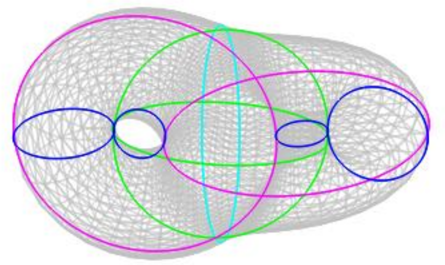
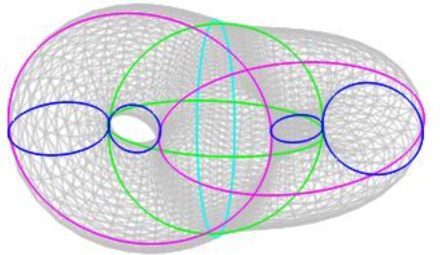


c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

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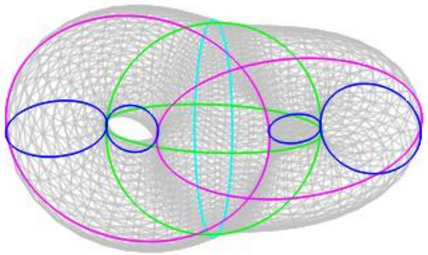

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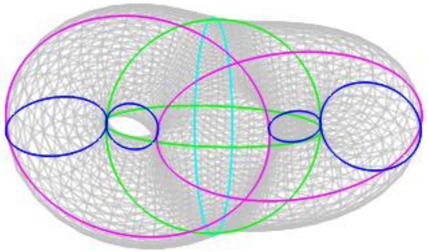

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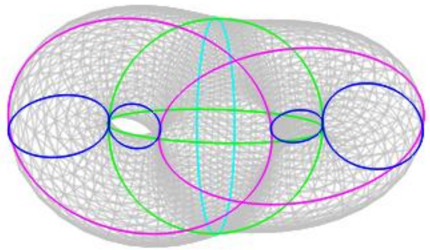

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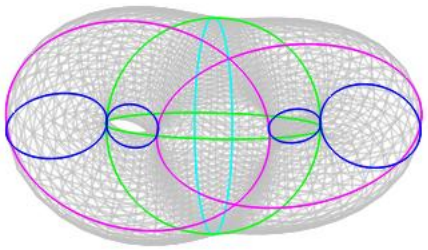

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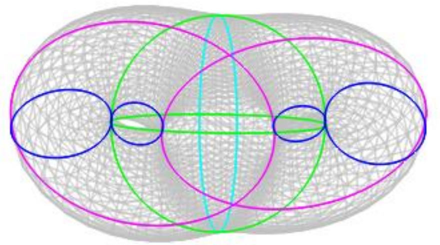

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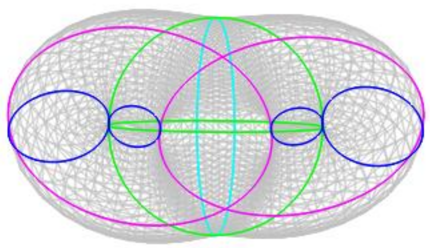

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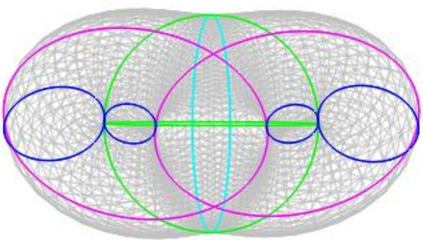

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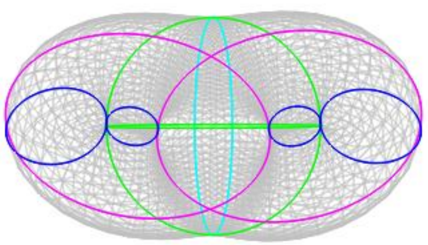

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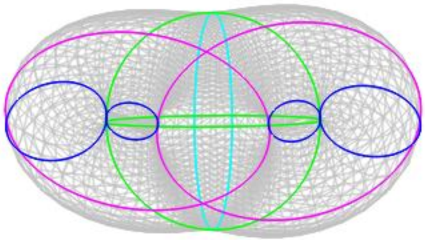

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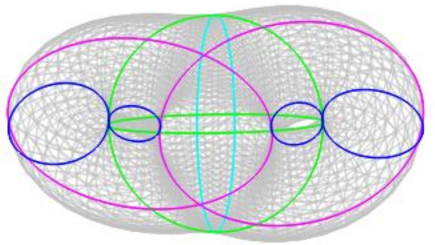
A curve system for EX^*



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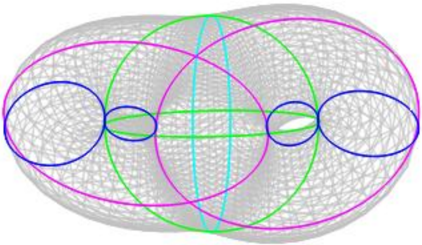
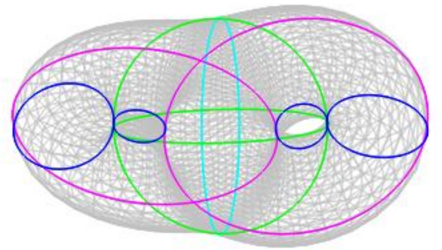
A curve system for EX^*



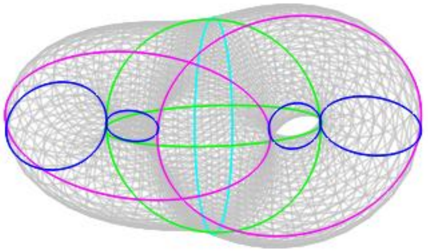
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



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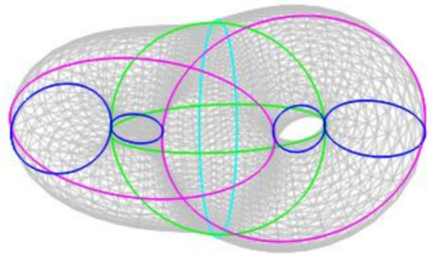
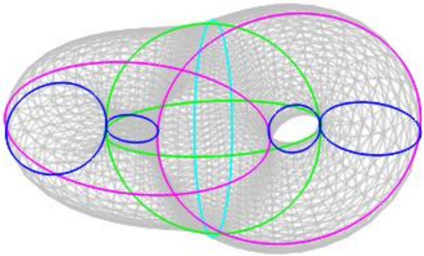
A curve system for EX^*



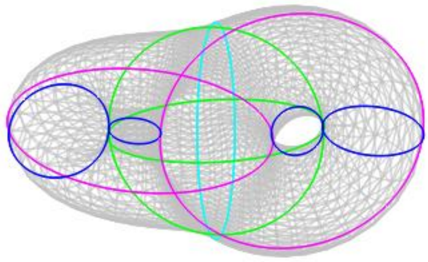
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



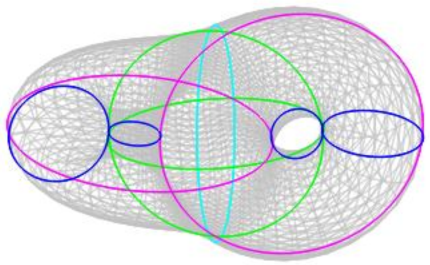
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



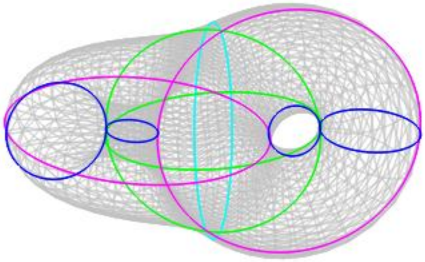
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



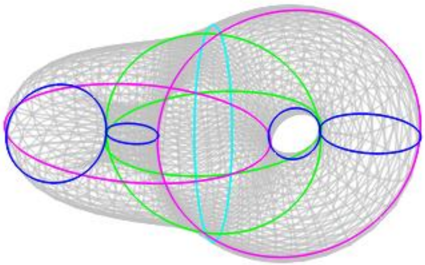
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



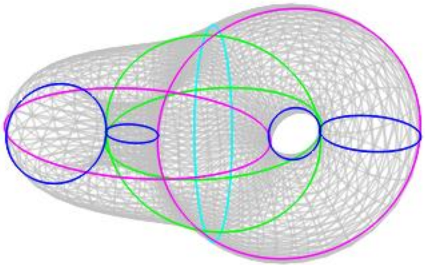
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



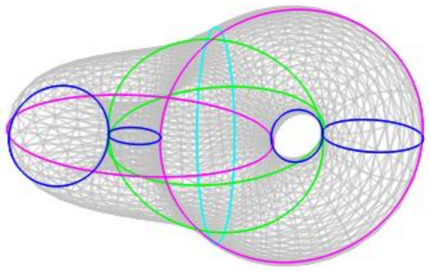
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



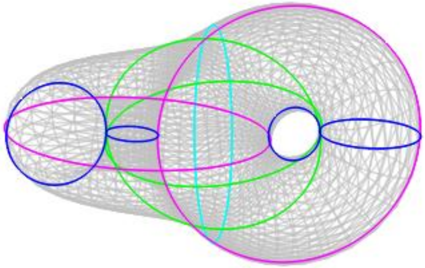
$c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$

A curve system for EX^*



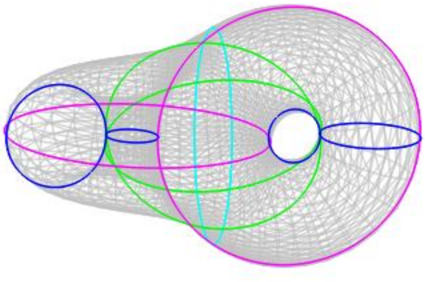
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



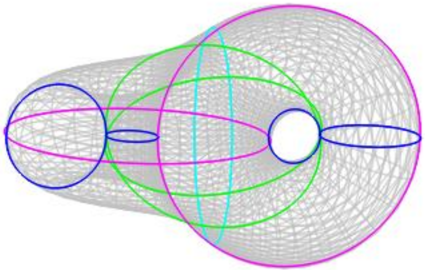
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



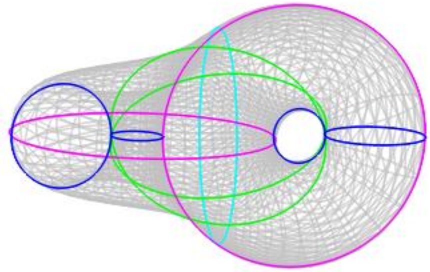
c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for EX^*



c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8

A curve system for $PX(a)$

Put $d(w,x,y) = (w/x^3, x/y)$; define $c_0, \dots, c_8 \colon \mathbb{R}/2\pi\mathbb{Z} \to PX(a)$ as follows:

$$c_0(t) = d(-\sqrt{a^{-2} + a^2 - 2\cos(4t)},\ e^{it},\ e^{-it})$$

$$c_1(t) = d\left(\frac{1+i}{8\sqrt{2}}\sin(t)\sqrt{16\cos(t)^2 + (a+a^{-1})^2\sin(t)^4},\ \frac{1+\cos(t)}{2},\ \frac{1-\cos(t)}{2}i\right)$$

$$c_2(t) = \lambda(c_1(t))$$

$$c_3(t) = d\left(-i\frac{a^{-1}-a}{8}\sin(t)\sqrt{(1+a)^4 - (1-a)^4\cos(t)^2}\sqrt{(1+a)^2 - (1-a)^2\cos(t)^2},\right.$$
$$\left.\frac{(1+a)+(1-a)\cos(t)}{2},\ \frac{(1+a)-(1-a)\cos(t)}{2}\right)$$

$$c_4(t) = \lambda(c_3(t))$$

$$c_5(t) = \left(\frac{\sin(t)}{8}\sqrt{2a(3-\cos(t))(4-a^4(1-\cos(t))^2)},\ a\frac{1-\cos(t)}{2}\right)$$

$$c_6(t) = \lambda(c_5(t)), \qquad c_7(t) = \mu(c_5(t)), \qquad c_8(t) = \lambda\mu(c_5(t)).$$

These give a curve system.

Put $d(w,x,y) = (w/x^3, x/y)$; define $c_0, \ldots, c_8 \colon \mathbb{R}/2\pi\mathbb{Z} \to PX(a)$ as follows:

$$c_0(t) = d(-\sqrt{a^{-2} + a^2 - 2\cos(4t)},\ e^{it},\ e^{-it})$$

$$c_1(t) = d\left(\frac{1+i}{8\sqrt{2}} \sin(t)\sqrt{16\cos(t)^2 + (a + a^{-1})^2 \sin(t)^4},\ \frac{1+\cos(t)}{2},\ \frac{1-\cos(t)}{2} i\right)$$

$$c_2(t) = \lambda(c_1(t))$$

$$c_3(t) = d\left(-i\frac{a^{-1} - a}{8} \sin(t)\sqrt{(1+a)^4 - (1-a)^4\cos(t)^2}\sqrt{(1+a)^2 - (1-a)^2\cos(t)^2},\right.$$
$$\left.\frac{(1+a)+(1-a)\cos(t)}{2},\ \frac{(1+a)-(1-a)\cos(t)}{2}\right)$$

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These give a curve system.

When $|m| > 1$ with $d = \sqrt{|m|^2 - 1}$ we have a geodesic $\omega_m \colon \mathbb{R} \to \Delta$:

$$\omega_m(s) = \frac{id - 1}{\overline{m}} \frac{(id+1)e^{-s} - i|m|e^s}{i|m|e^s + (id-1)e^{-s}}.$$

Put

$$s_0 = 2\log\left(\frac{\sqrt{2}a}{a_+ - a_-}\right) \qquad s_2 = \log\left(\frac{1+a}{a_-}\right) \qquad s_4 = \frac{1}{4}\log\left(\frac{a_+^2 + 2a_+ + 2}{a_+^2 - 2a_+ + 2}\right)$$

$$s_1 = \frac{1}{2}\log\left(\frac{\sqrt{2} + a_+}{\sqrt{2} - a_+}\right) \qquad s_3 = \frac{1}{2}\log\left(\frac{a + a_+ + 1}{a + a_+ - 1}\right)$$

We then define maps $\widetilde{c}_k \colon \mathbb{R} \to \Delta$ for $0 \le k \le 8$ as follows:

$$\widetilde{c}_0(t) = \omega_{(1+i)/a_+}((t/\pi - 1/4)s_0) \qquad \widetilde{c}_5(t) = \tanh(t\, s_3/\pi)$$

$$\widetilde{c}_1(t) = e^{i\pi/4}\tanh(t\, s_1/\pi) \qquad \widetilde{c}_6(t) = i\,\tanh(t\, s_3/\pi)$$

$$\widetilde{c}_2(t) = e^{3i\pi/4}\tanh(t\, s_1/\pi) \qquad \widetilde{c}_7(t) = \omega_{ia_+/2+1/a_+}(t\, s_3/\pi - s_4)$$

$$\widetilde{c}_3(t) = \omega_{a_+}(-t\, s_2/\pi) \qquad \widetilde{c}_8(t) = \omega_{a_+/2+i/a_+}(-t\, s_3/\pi + s_4)$$

$$\widetilde{c}_4(t) = \omega_{ia_+}(-t\, s_2/\pi)$$

This gives a curve system on $HX(a)$.

A curve system for $HX(a)$

When $|m| > 1$ with $d = \sqrt{|m|^2 - 1}$ we have a geodesic $\omega_m : \mathbb{R} \to \Delta$:

$$\omega_m(s) = \frac{id - 1}{\overline{m}} \frac{(id+1)e^{-s} - i|m|e^s}{i|m|e^s + (id-1)e^{-s}}.$$

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$$s_1 = \frac{1}{2} \log \left(\frac{\sqrt{2} + a_+}{\sqrt{2} - a_+} \right) \qquad s_3 = \frac{1}{2} \log \left(\frac{a + a_+ + 1}{a + a_+ - 1} \right)$$

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$$s_1 = \frac{1}{2}\log\left(\frac{\sqrt{2}+a_+}{\sqrt{2}-a_+}\right) \qquad s_3 = \frac{1}{2}\log\left(\frac{a+a_+ + 1}{a+a_+ - 1}\right)$$

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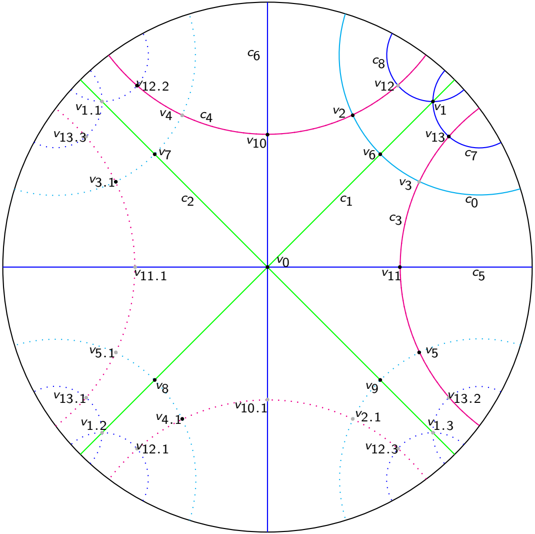
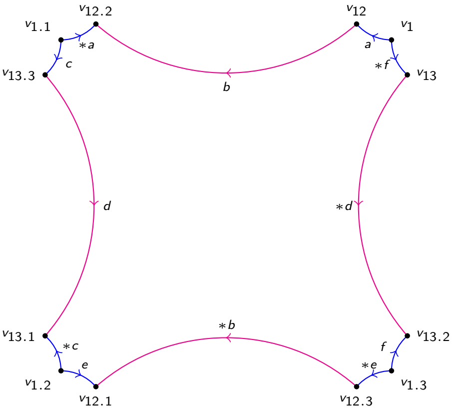
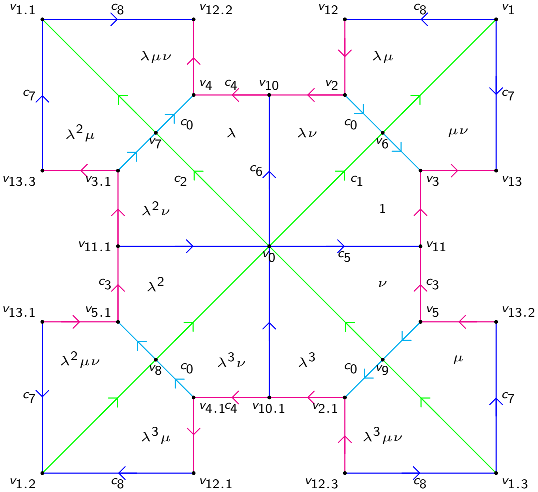
This gives a curve system on $HX(a)$.

Pictures for $HX(a)$




$v_{12.2}$
v_{12}
$v_{1.1}$
v_1
$*a$
a
c
$*f$
$v_{13.3}$
v_{13}
b
d
$*d$
$*b$
$v_{13.1}$
$v_{13.2}$
$*c$
f
e
$*e$
$v_{1.2}$
$v_{1.3}$
$v_{12.1}$
$v_{12.3}$

Fundamental domains and nets



Any cromulent surface has a net as shown above. Each of the 16 regions is a fundamental domain for the action of G.

Alternative nets

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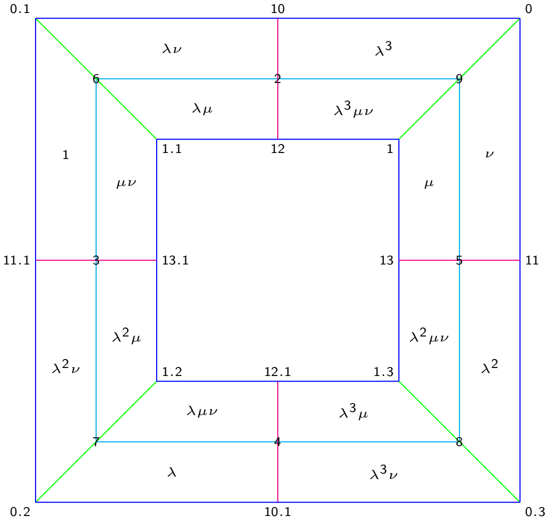

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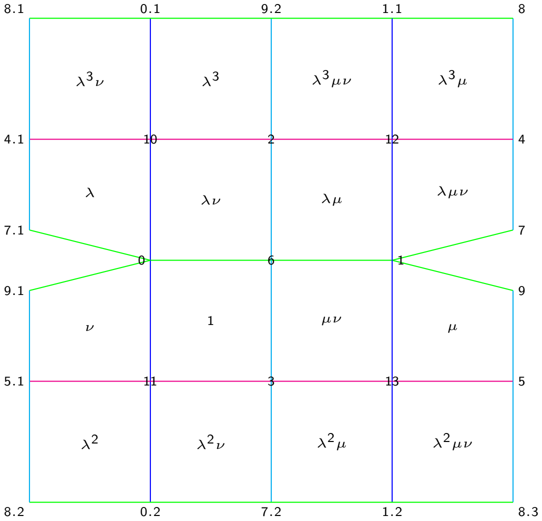

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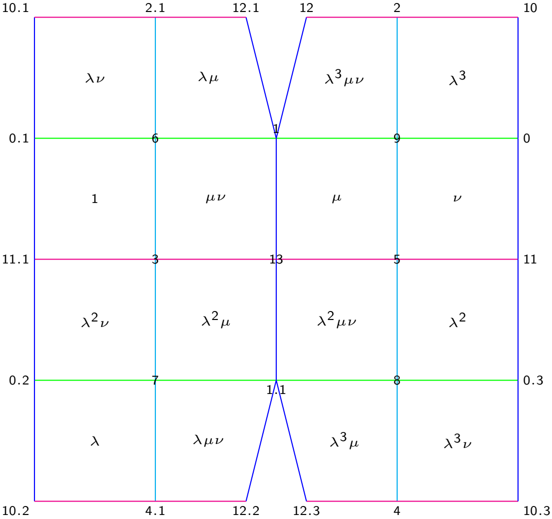

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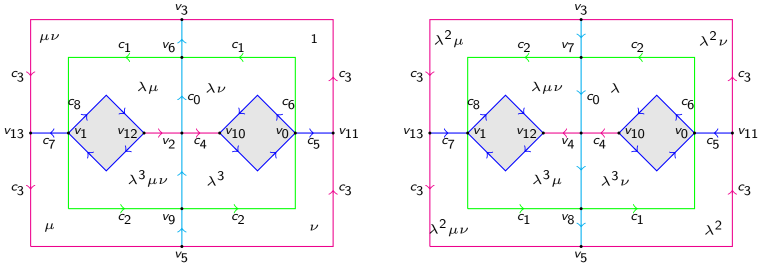
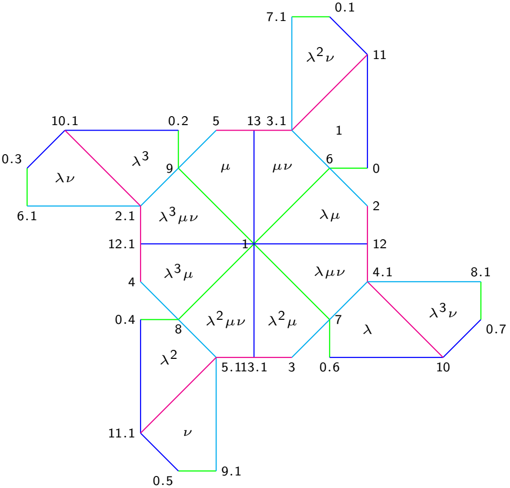


This gives a "pair of pants" decomposition.

Alternative nets



This gives a presentation of π_1 as

$$\Pi = \langle \beta_i \mid i \in \mathbb{Z}/8 \rangle / \langle \beta_i \beta_{i+4}, \beta_0\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7 \rangle$$

Homology

Let X be a cromulent surface. Then there is an isomorphism $\psi\colon H_1(X) \to \mathbb{Z}^4$, with the following effect on the homology classes of the curves c_k:

$$\begin{aligned}
\psi(c_0) &= (0, 0, 0, 0) \\
\psi(c_1) &= (1, 1, -1, -1) & \psi(c_2) &= (-1, 1, 1, -1) \\
\psi(c_3) &= (0, 1, 0, -1) & \psi(c_4) &= (-1, 0, 1, 0) \\
\psi(c_5) &= (1, 0, 0, 0) & \psi(c_6) &= (0, 1, 0, 0) \\
\psi(c_7) &= (0, 0, 1, 0) & \psi(c_8) &= (0, 0, 0, 1).
\end{aligned}$$

This is equivariant with respect to the following action of G on $\mathbb{Z}^4$:

$$\begin{aligned}
\lambda(n) &= (-n_2, n_1, -n_4, n_3) \\
\mu(n) &= (n_3, -n_4, n_1, -n_2) \\
\nu(n) &= (n_1, -n_2, n_3, -n_4).
\end{aligned}$$

Moreover, the intersection product on $H_1(X)$ corresponds to the following bilinear form on $\mathbb{Z}^4$:

$$(n, m) = n_1 m_2 - n_2 m_1 - n_3 m_4 + n_4 m_3.$$

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Quotients

- If X is cromulent and $H \leq \langle \lambda, \mu \rangle$ then X/H is a compact Riemann surface.
- The study of these quotients is essentially the same as the Galois theory of the field of rational functions on X.
- If $H = 1$ then $X/H = X$; if $H = \{1, \lambda^i \mu\}$ for some i then X/H is an elliptic curve; in all other cases $X/H \simeq \mathbb{C}_\infty$.
- The elliptic cases are the most interesting and important.
- In the case $X = PX(a)$, we can write explicit formulae for everything, involving elliptic integrals and the Weierstrass $\wp$-function in appropriate places.

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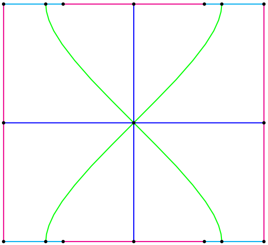

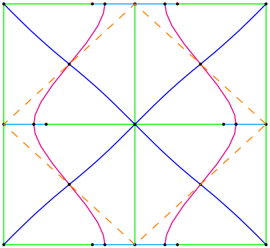

Relating the projective and hyperbolic families

The key problem is to understand the map

$$p = (\Delta \to \Delta/\Pi = HX(b) \xrightarrow{\simeq} PX(a) \to PX(a)/\langle \lambda^2 \rangle \xrightarrow{\simeq} \mathbb{C}_\infty),$$

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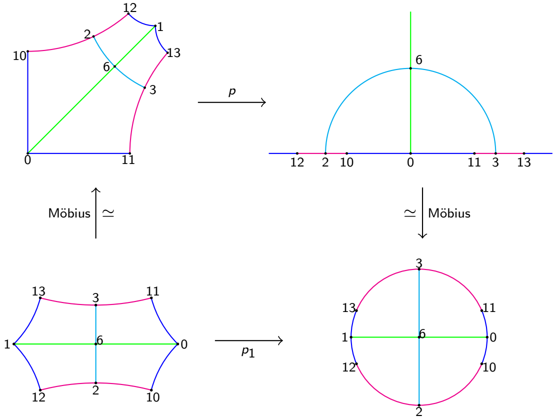
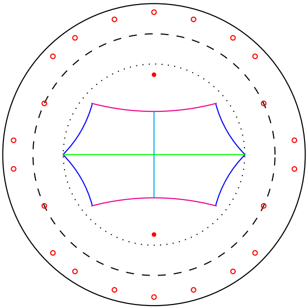


Equivariance properties of p imply that $p_1(z)$ is odd, with real Taylor coefficients, and that the poles are as follows:



The known behaviour of p at v_0, v_3 and v_{11} gives further constraints on the general form of $p_1(z)$. We can then use numerical methods to find coefficients such that p_1 sends the blue and magenta arcs above to the unit circle.

The Schwarzian derivative operator is $S(f) = f'''/f' - \frac{3}{2}(f''/f')^2$.

Proposition: $S(p_1^{-1}) = s_0^* + ds_1^*$, where d is a real constant and

$$s_0^*(z) = \frac{192a^4z^2(1+z^2)^2 - 9(1-a^4)^2(1-z^2)^4}{2(1-z^2)^2((1+a^2)^2(1-z^2)^2+16a^2z^2)^2} \qquad s_1^*(z) = \frac{4a^2}{(1+a^2)^2(1-z^2)^2+16a^2z^2}.$$

Proof: We can define $d = (S(p_1^{-1}) - s_0^*)/s_1^*$; then d is a meromorphic function on $\mathbb{C}_\infty$, and we need to show that it is constant, or equivalently, that it is holomorphic. The equivariance properties of p determine the branching behaviour of p_1^{-1}, and this in turn determines the poles of $S(p_1^{-1})$. Using this we can see that d has no poles. $\square$

There is a classical theory which relates solutions of the nonlinear equation $S(f) = s$ to solutions of the linear equation $2g'' + sg = 0$. If we knew d, this would allow us to find p_1^{-1} ad thus p_1. In practice we have to guess d, find p_1, and then repeatedly adjust d to eliminate inconsistencies.

(This method is good if we start by knowing a; the earlier method is better if we start by knowing b.)

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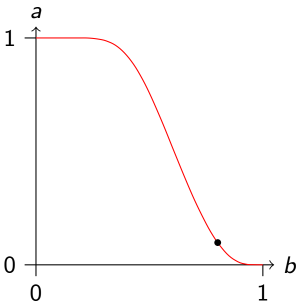
The graph of a against b

This is the graph of $a = a_P$ against $b = a_H$:



It is very flat at $b = 1$, and even flatter at $b = 0$. The marked point indicates the values that are relevant for EX^*.

Polynomial functions on EX^*

Let A be the ring of polynomial functions on EX^*. We put

$$
\begin{aligned}
y_1 &= x_3 & y_2 &= (x_2^2 - x_1^2)/\sqrt{2} - \tfrac{3}{2}x_3x_4 \\
z_1 &= y_1^2 & z_2 &= y_2^2 \\
u_1 &= \tfrac{1}{2}(1 - \sqrt{2}y_2)(1 - y_1^2(1 - y_2/\sqrt{2})) & u_2 &= \tfrac{1}{2}(1 + \sqrt{2}y_2)(1 - y_1^2(1 + y_2/\sqrt{2}))
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Proposition:

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$$
\begin{aligned}
y_1 &= x_3 & y_2 &= (x_2^2 - x_1^2)/\sqrt{2} - \tfrac{3}{2}x_3x_4 \\
z_1 &= y_1^2 & z_2 &= y_2^2 \\
u_1 &= \tfrac{1}{2}(1-\sqrt{2}y_2)(1-y_1^2(1-y_2/\sqrt{2})) & u_2 &= \tfrac{1}{2}(1+\sqrt{2}y_2)(1-y_1^2(1+y_2/\sqrt{2}))
\end{aligned}
$$

$$
\begin{aligned}
\lambda^*(x_1) &= -x_2 & \mu^*(x_1) &= x_1 & \nu^*(x_1) &= x_1 \\
\lambda^*(x_2) &= x_1 & \mu^*(x_2) &= -x_2 & \nu^*(x_2) &= -x_2 \\
\lambda^*(x_3) &= x_3 & \mu^*(x_3) &= -x_3 & \nu^*(x_3) &= x_3 \\
\lambda^*(x_4) &= -x_4 & \mu^*(x_4) &= -x_4 & \nu^*(x_4) &= x_4 \\
\lambda^*(y_1) &= y_1 & \mu^*(y_1) &= -y_1 & \nu^*(y_1) &= y_1 \\
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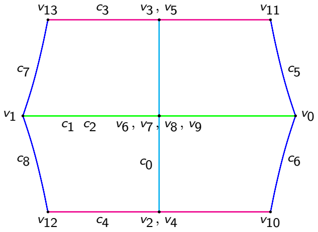
\lambda^*(z_1) &= z_1 & \mu^*(z_1) &= z_1 & \nu^*(z_1) &= z_1 \\
\lambda^*(z_2) &= z_2 & \mu^*(z_2) &= z_2 & \nu^*(z_2) &= z_2 .
\end{aligned}
$$

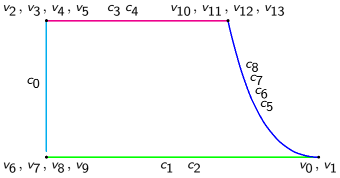
Proposition:

- $A^{\langle \lambda^2, \nu \rangle} = \mathbb{R}[y_1, y_2]$, and $A^G = \mathbb{R}[z_1, z_2]$.
- $A = \mathbb{R}[y_1, y_2]\{1, x_1, x_2, x_1x_2\} = \mathbb{R}[y_1, y_2][x_1, x_2]/(x_i^2 - u_i)$.

The y-plane and the z-plane

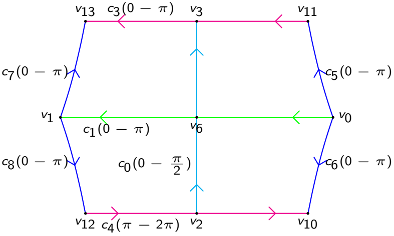


$EX^* / \langle \lambda^2, \nu \rangle$, with coordinates (y_1, y_2)



EX^* / G, with coordinates (z_1, z_2)

The y-plane and the z-plane



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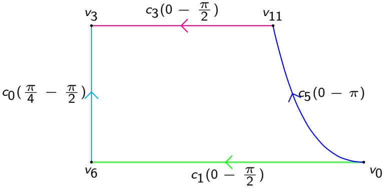

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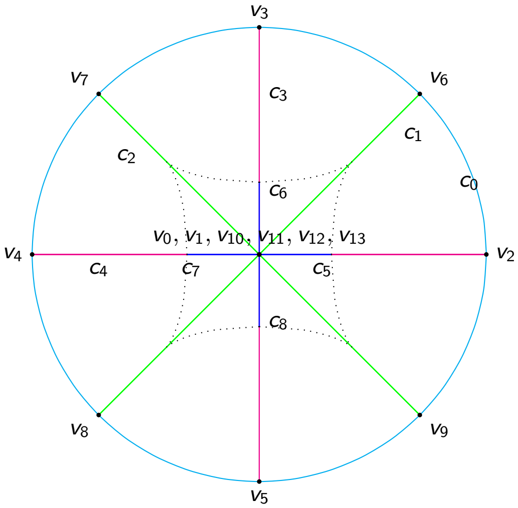
Some linear projections



$$\pi(x) = (x_1, x_2)$$

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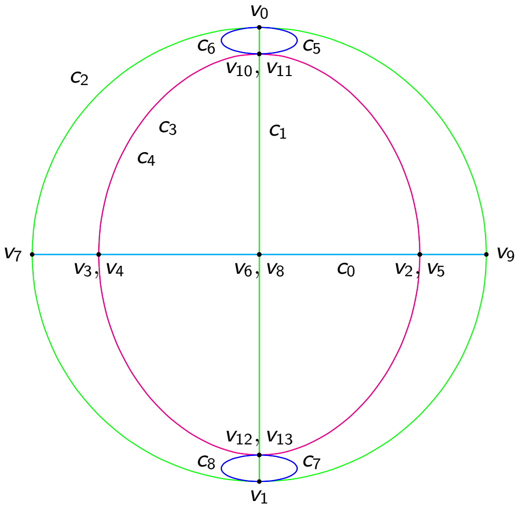

$$\delta(x) = ((x_1 - x_2)/\sqrt{2}, x_3)$$

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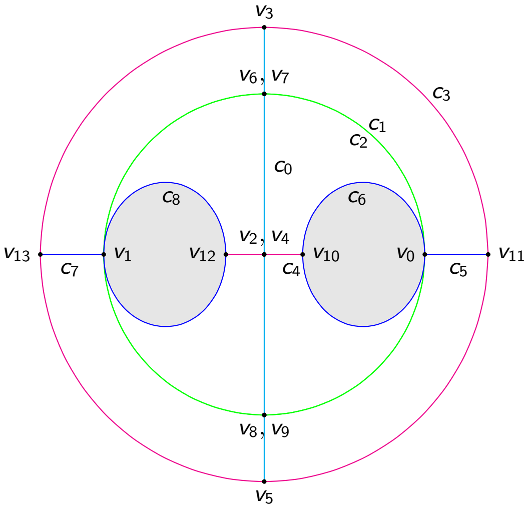

$$\zeta(x) = ((x_3 - x_4)/\sqrt{2}, x_2)$$

Barycentric coordinates and integration

For $x \in EX^*$, let T_x be the tangent plane in $\mathbb{R}^4$, shifted so that $x \in T_x$. Let $\pi_x : \mathbb{R}^4 \to T_x$ be the orthogonal projection.

Suppose $a_0, a_1, a_2, x \in EX^*$ are close together. Then there will be a unique $t \in \mathbb{R}^3$ with $\sum_i t_i = 1$ and $x = \sum_i t_i \pi_x(a_i)$. These are *barycentric coordinates* for x relative to $\underline{a}$. We write $T(\underline{a})$ for the set where all t_i are nonnegative; this is a triangle.

It works out that barycentric coordinates relative to (a_0, a_1, a_2) and (a_0, a_1, a_3) agree on the edge joining a_0 and a_1. Because of this, we can use barycentric coordinates to triangulate EX^*.

There is a nice formula for the barycentric coordinate map and its Jacobian. Because of this, we can use a barycentric triangulation to calculate integrals over EX^*.

This method of integration is slow. However, we can use it to integrate polynomials, then find more efficient quadrature rules that do the right thing for polynomials.

Accuracy can be tested using Gauss-Bonet and Stokes.

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Proposition: The Gaussian curvature the standard metric m on EX^* is

$$K(m) = K_0 = 1 + 8\frac{2z_2 - 1}{(2 - z_1)^2(1 + z_2)^2}.$$

For any $f\colon EX^* \to \mathbb{R}$, we also have

$$K(e^{2f}m) = (K_0 - \Delta(f))/e^{2f}$$

Proposition: The Laplacian is given by

$$\Delta(f) = \sum_i \frac{\partial^2 f}{\partial x_i^2} - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{1}{r^2}\sum_{i,j} n_i n_j \frac{\partial^2 f}{\partial x_i \partial x_j} - 2\sum_i x_i \frac{\partial f}{\partial x_i} + \left(\frac{r''}{r^4} - \frac{r'}{r^2}\right)\sum_i n_i \frac{\partial f}{\partial x_i}$$

(where n, r, r' and r'' are given by simple formulae in terms of x_i).
If f is G-invariant, then the formula can be rewritten in terms of z_1 and z_2.

Proposition: There is a unique smooth f such that the metric $m_{\text{hyp}} = e^{2f}m$ has $K(m_{\text{hyp}}) = -1$. For this metric, the holomorphic covering map $\Delta \to EX^*$ is isometric.

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Uniformizing EX^*

We first want to find f such that $K(e^{2f}m) = -1$. We let F be the space of rational functions p/q, where p and q are polynomial of degree at most 8 in (z_1, z_2). We search numerically for $f \in F$ minimizing $\int_{EX^*}(K(e^{2f}m)+1)^2$.

Now if $EX^* \simeq HX(b)$, then the length of the curve $C_k \subset EX^*$ with respect to $e^{2f}m$ should be given by a known formula in terms of b. Each k gives an estimate for b; these differ by about $10^{-7.4}$.

At any point in EX^*, we can use power series methods to find an approximate conformal chart, then modify it to make it approximately isometric for the hyperbolic metrics on Δ and EX^*. Any two such charts should be related by an isometry of Δ, which must be $z \mapsto \lambda(z-\alpha)/(1-\overline{\alpha}z)$ with $|\lambda| = 1 > |\alpha|$.

We use numerical methods to line up a large number of such charts as accurately as possible. This enables us to compute the canonical covering map $q\colon \Delta \to EX^* \subset \mathbb{R}^4$ at many points.

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$$q_1(re^{i\theta}) = \sum_m a_{1,m}(r)\cos((2m+1)\theta) \quad q_2(re^{i\theta}) = \sum_m (-1)^m a_{1,m}(r)\sin((2m+1)\theta)$$

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The functions $a_{k,m}(r)$ can be represented accurately by splines, but not by polynomials or rational functions. We do not yet know a more theoretically illuminating approach.

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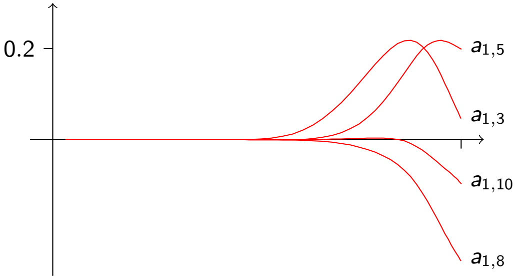

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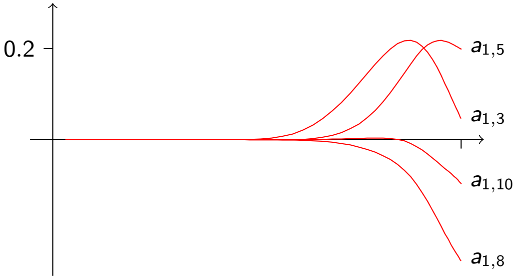
$$q_3(re^{i\theta}) = \sum_m a_{3,m}(r)\cos(4m\theta) \qquad q_4(re^{i\theta}) = \sum_m a_{4,m}(r)\cos((4m+2)\theta).$$



The functions $a_{k,m}(r)$ can be represented accurately by splines, but not by polynomials or rational functions. We do not yet know a more theoretically illuminating approach.