Symmetric Powers of Spheres

Neil Strickland
(with Johann Sigurdsson)

August 9, 2007

$\pi_*(S^*)$

Overview of homotopy theory

$$\pi_*(S^*) \longrightarrow H_*(S^0)$$

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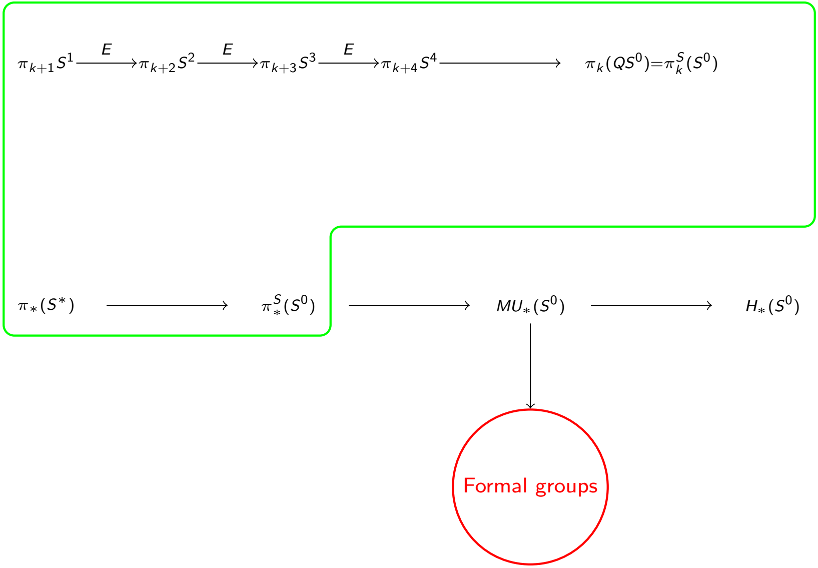
$$MU_*(S^0) \big\downarrow$$

Formal groups

Overview of homotopy theory



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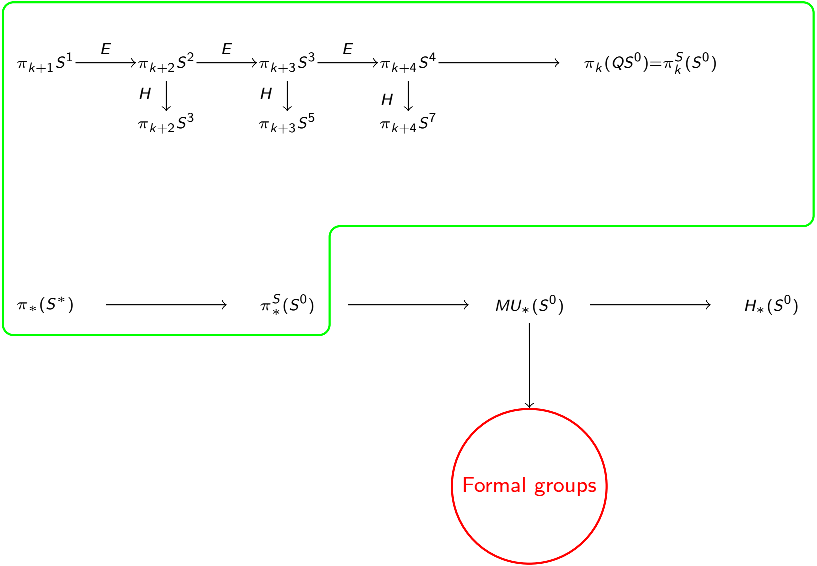

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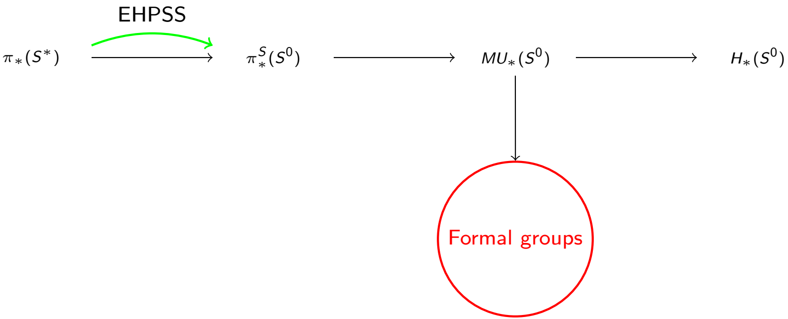

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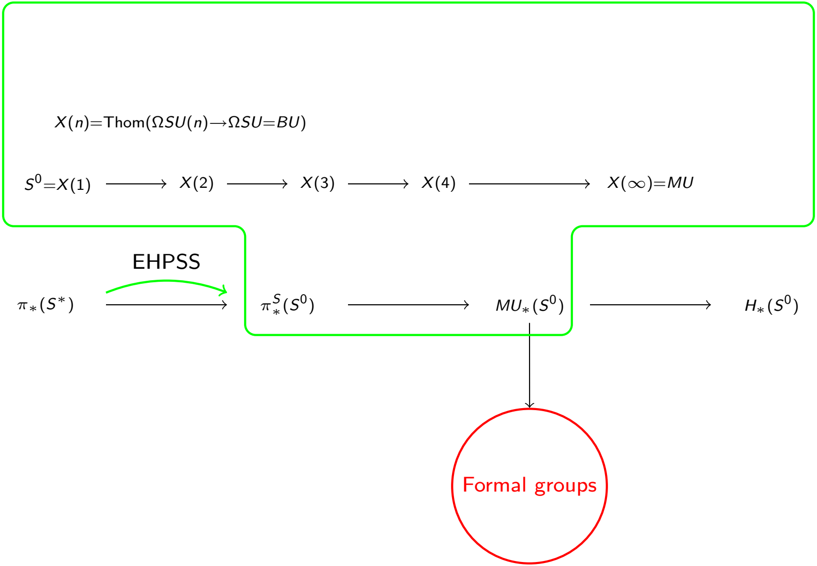

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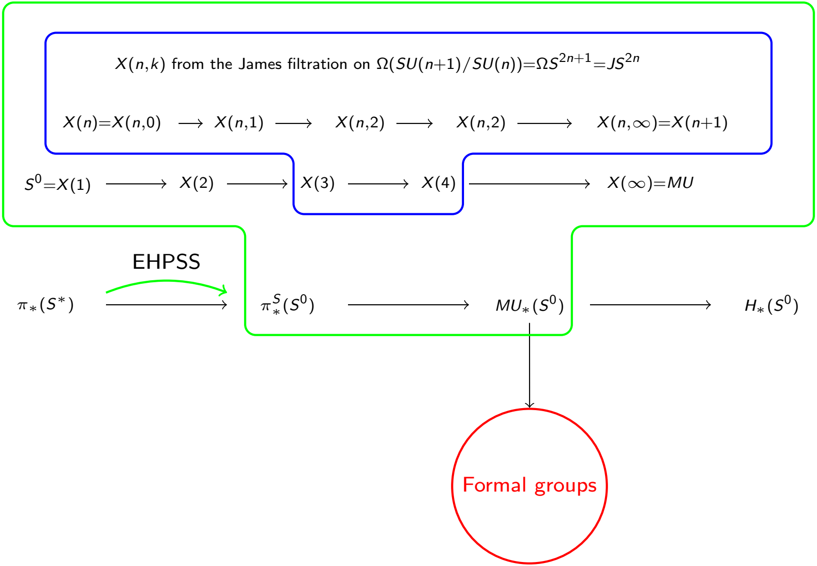

Overview of homotopy theory

EHPSS   Nilpotence filtration

$$\pi_*(S^*) \longrightarrow \pi_*^S(S^0) \longrightarrow MU_*(S^0) \longrightarrow H_*(S^0)$$

Overview of homotopy theory

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Overview of homotopy theory

$$S^0 = \mathrm{SP}^1(S^0) \longrightarrow \mathrm{SP}^2(S^0) \longrightarrow \mathrm{SP}^3(S^0) \longrightarrow \mathrm{SP}^4(S^0) \longrightarrow \mathrm{SP}^\infty(S^0) = H$$

$\mathrm{SP}^n(S^0)$= prespectrum with k'th space $(S^k)^{\times n}/\Sigma_n$

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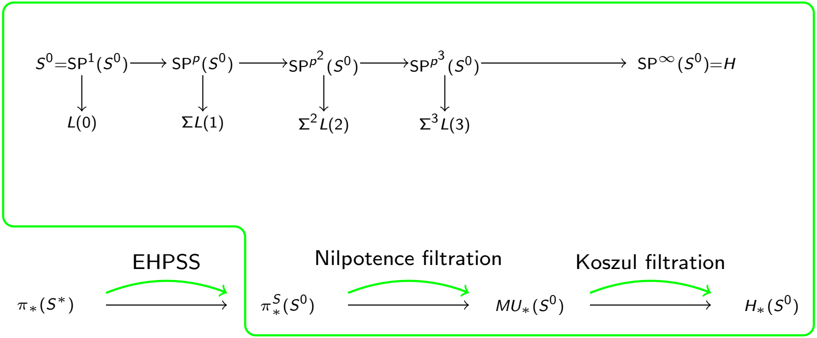

Overview of homotopy theory

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\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
L(0) & & \Sigma L(1) & & \Sigma^2 L(2) & & \Sigma^3 L(3) & &
\end{array}
$$

$\Omega^\infty L(*)$ is a DGA up to homotopy, chain equivalent to $\mathbb{Z}$ (Whitehead, Kuhn, Priddy)

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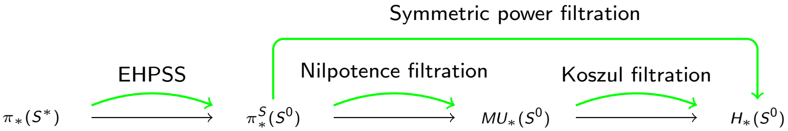

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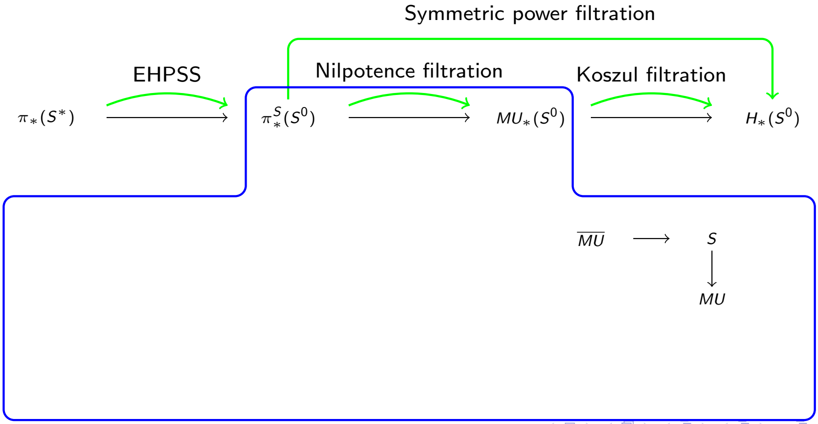

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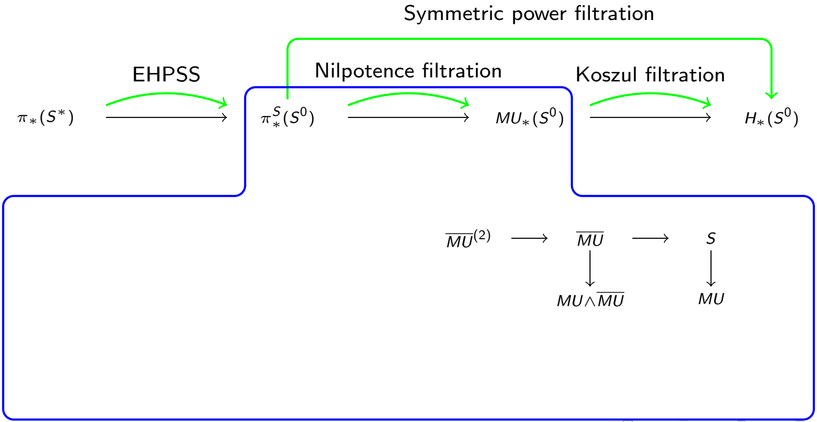

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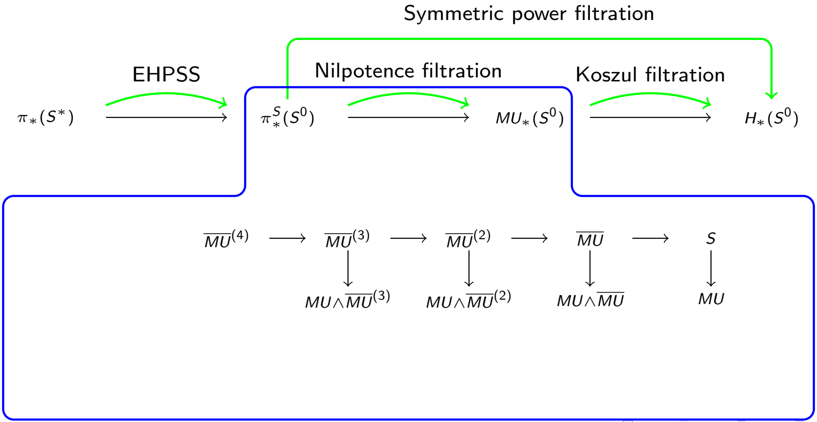

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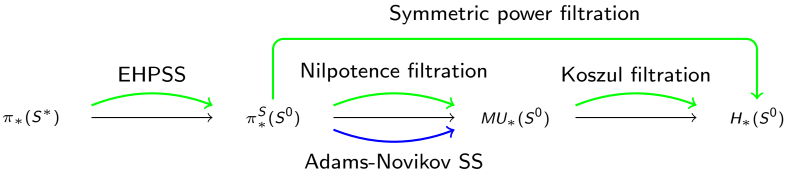

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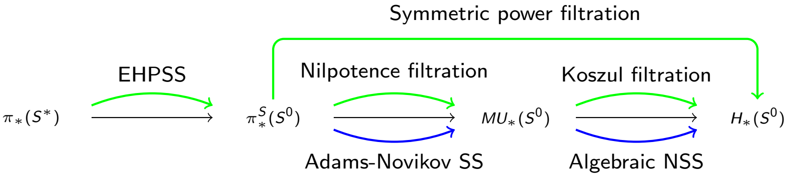

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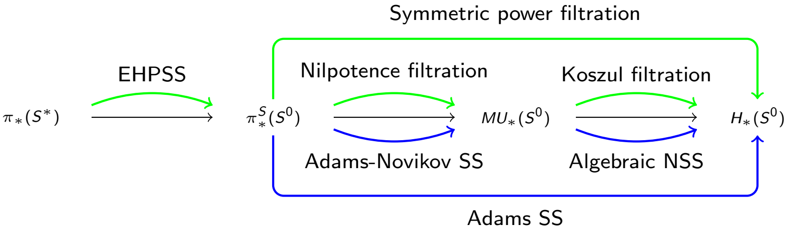

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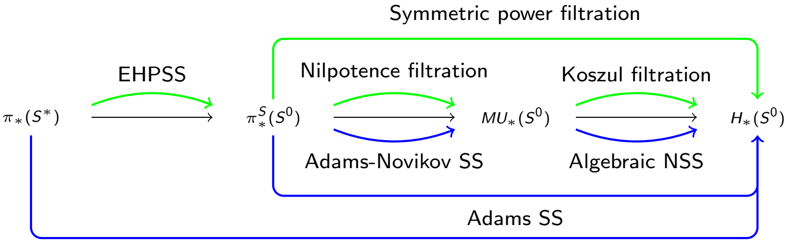
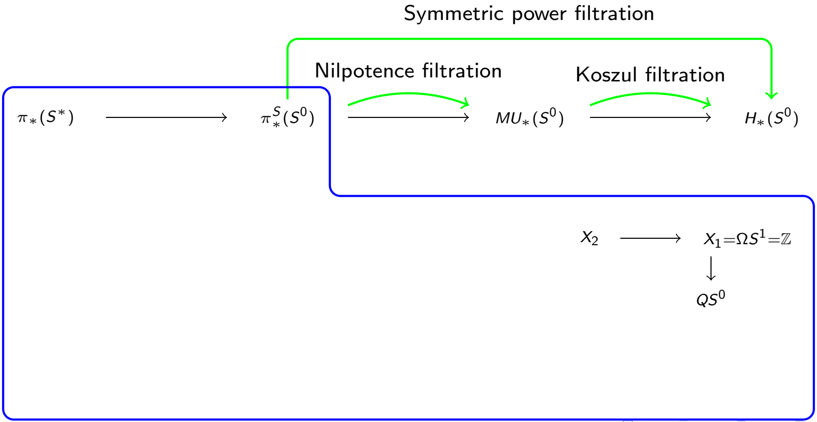


Unstable Adams SS, Lambda algebra, central series for simplicial groups

Overview of homotopy theory



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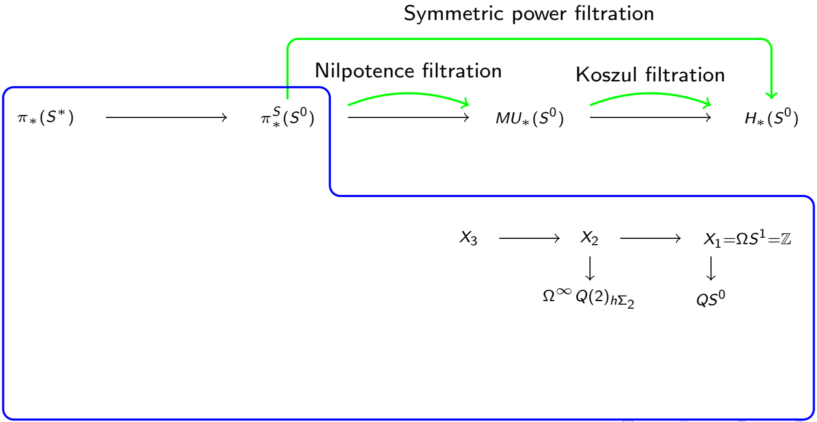

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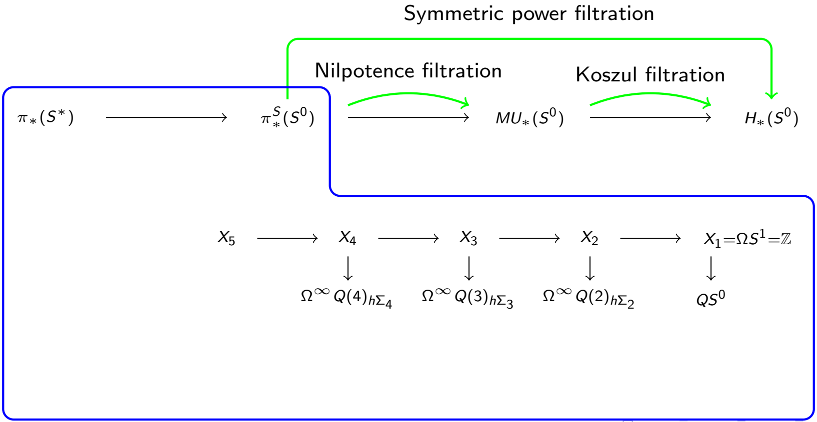

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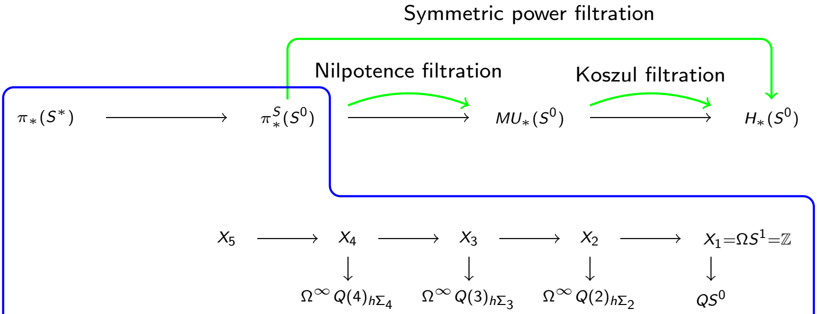

$Q(n)$ is a certain finite Σ_n-spectrum, with $H_* Q(n) = H_0 Q(n) = \text{Lie}(n)$.

Overview of homotopy theory

Symmetric power filtration

Nilpotence filtration

Koszul filtration

$$\pi_*(S^*) \longrightarrow \pi_*^S(S^0) \longrightarrow MU_*(S^0) \longrightarrow H_*(S^0)$$

$$\begin{array}{ccccccccc}
X_5 & \longrightarrow & X_4 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 = \Omega S^1 = \mathbb{Z} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
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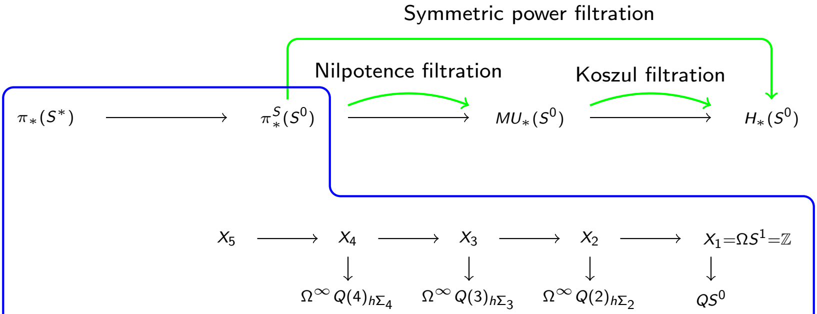

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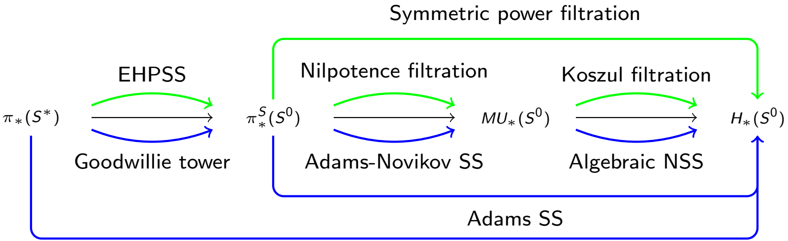
(Goodwillie, Johnson, Arone, Mahowald)

Overview of homotopy theory



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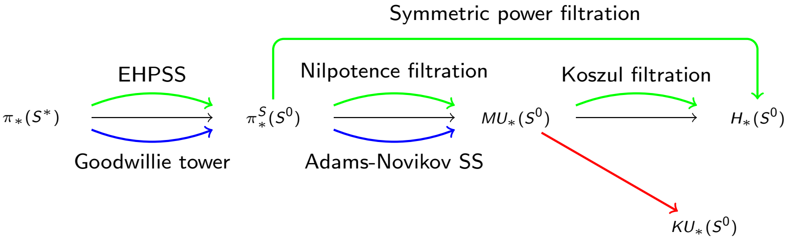

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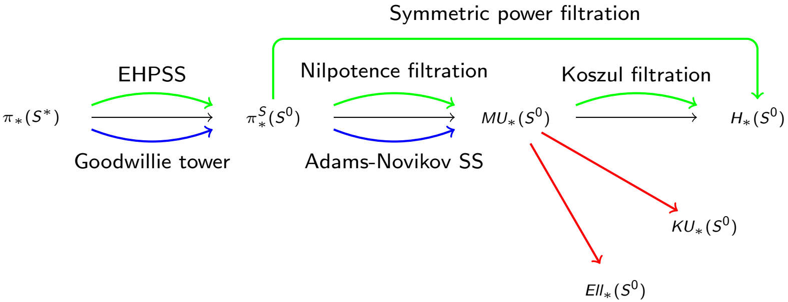

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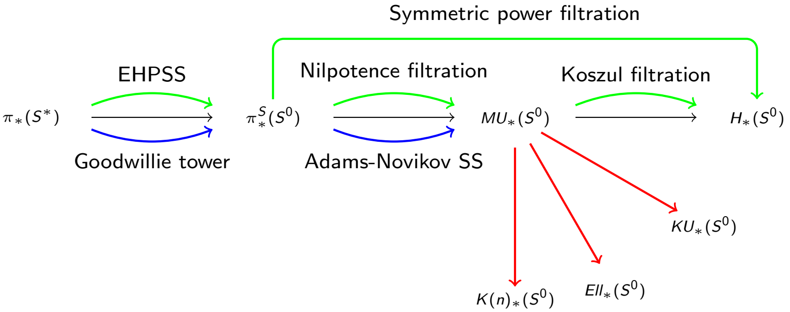

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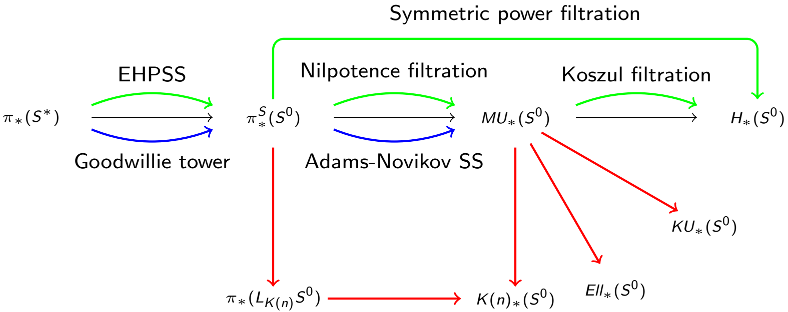

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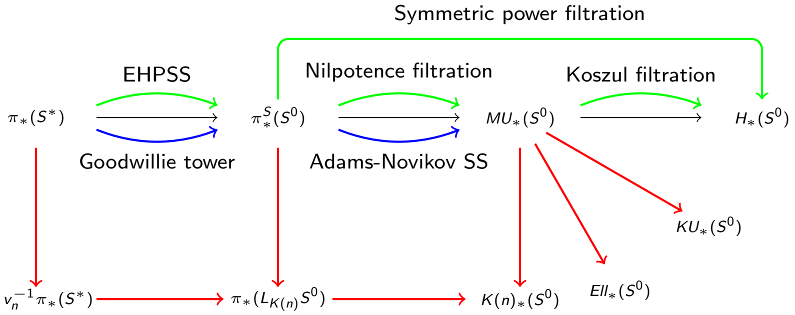

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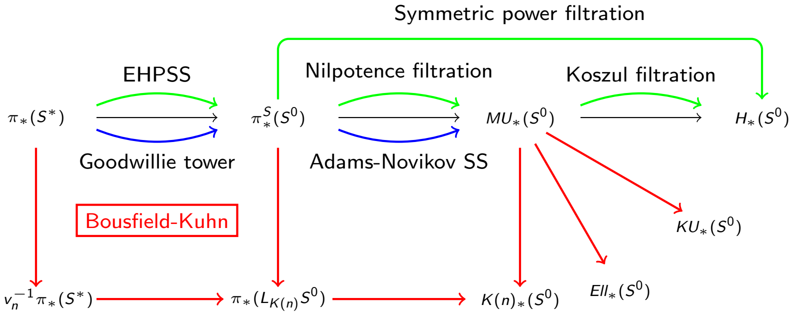

$\Phi_n\colon \text{Spaces} \to \text{Spectra} \qquad \Phi_n(\Omega^\infty X) = L_{K(n)}(X)$

Symmetric powers of unstable spheres

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\end{aligned}$$

$$\mathrm{SP}^n(S^1) = S^1 \qquad \mathrm{SP}^n(S^2) = \mathcal{P}^n$$

$$\overline{\mathrm{SP}}^n(S^1) = 0 \qquad \overline{\mathrm{SP}}^n(S^2) = S^{2n}$$

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There are natural product maps $\mathrm{SP}^n(S^V) \times \mathrm{SP}^m(S^W) \to \mathrm{SP}^{nm}(S^{V\oplus W})$ and $\overline{\mathrm{SP}}^n(S^V) \wedge \overline{\mathrm{SP}}^m(S^W) \to \overline{\mathrm{SP}}^{nm}(S^{V\oplus W})$.

Nontransitive subgroups

Let $\mathcal{F}$ be a family of subgroups of a finite group G, closed under subconjugacy. Then there is a G-space $E\mathcal{F}$ with

$$E\mathcal{F}^H = \begin{cases} \text{contractible} & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F}. \end{cases}$$

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Take $\mathcal{P}_n = \{\text{nontransitive subgroups of } \Sigma_n\}$;
then $E\mathcal{P}_n = S(\infty W_n)$ and so $\overline{\mathrm{SP}}^n(S^0) = \widetilde{\Sigma} B\mathcal{P}_n$.

K-theory of multisets

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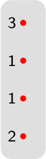
A multiset is a finite set with multiplicities.



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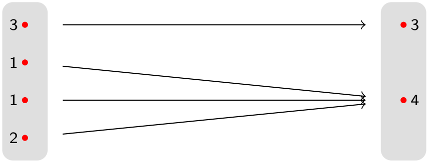
Morphisms are functions, bijective up to multiplicity.



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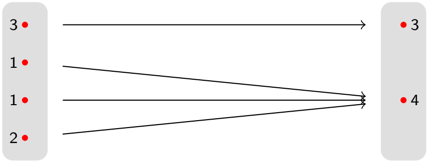

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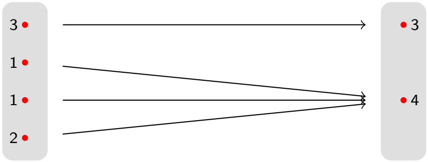

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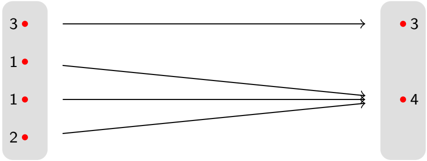

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Theorem(Lesh): $K(\mathcal{M}_n) = \mathrm{SP}^n(S^0)$ and $B\mathcal{M}_{n-1}^n = B\mathcal{P}_n$ and
$K(\mathcal{M}_n)/K(\mathcal{M}_{n-1}) = \overline{\mathrm{SP}}^n(S^0) = \widetilde{\Sigma} B\mathcal{M}_{n-1}^n$.

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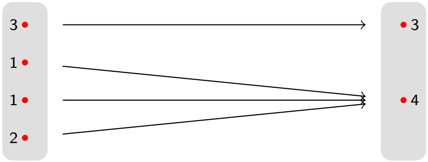
Morphisms are functions, bijective up to multiplicity.



$\mathcal{M} = \{\text{multisets}\}$ is symmetric bimonoidal under $\amalg$ and $\times$, so $K(\mathcal{M})$ is a ring spectrum. In fact $K(\mathcal{M}) = H$.
$\mathcal{M}_n$: maximum multiplicity $\leq n$; $\mathcal{M}^k$: total multiplicity k; $\mathcal{M}_n^k = \mathcal{M}_n \cap \mathcal{M}^k$
Theorem(Lesh): $K(\mathcal{M}_n) = \mathrm{SP}^n(S^0)$ and $B\mathcal{M}_{n-1}^n = B\mathcal{P}_n$ and
$K(\mathcal{M}_n)/K(\mathcal{M}_{n-1}) = \overline{\mathrm{SP}}^n(S^0) = \widetilde{\Sigma} B\mathcal{M}_{n-1}^n$.

$$\begin{array}{ccc} \mathrm{Free}(\mathcal{M}_{n-1}^n) & \longrightarrow & \mathcal{M}_{n-1} \\ \downarrow & & \downarrow \\ \mathrm{Free}(\mathcal{M}_n^n) & \longrightarrow & \mathcal{M}_n \end{array} \qquad \begin{array}{ccc} \Sigma_+^\infty B\mathcal{M}_{n-1}^n & \longrightarrow & K(\mathcal{M}_{n-1}) \\ \downarrow & & \downarrow \\ S^0 & \longrightarrow & K(\mathcal{M}_n) \end{array}$$

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There are still some open questions about how all this fits together, and how it dualises.

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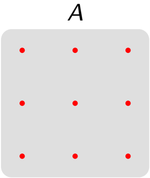

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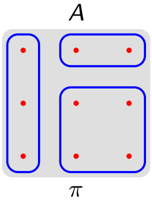

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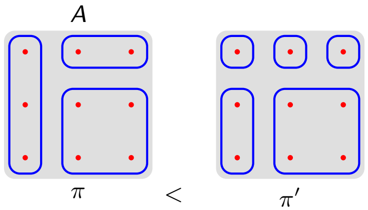

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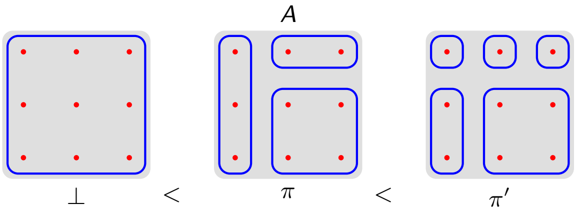

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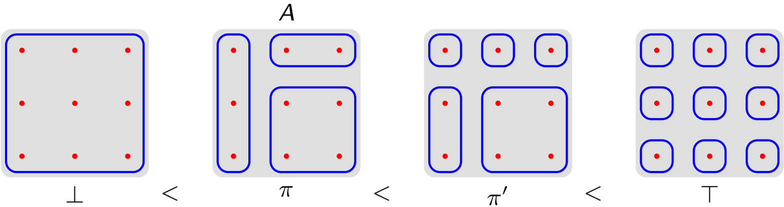

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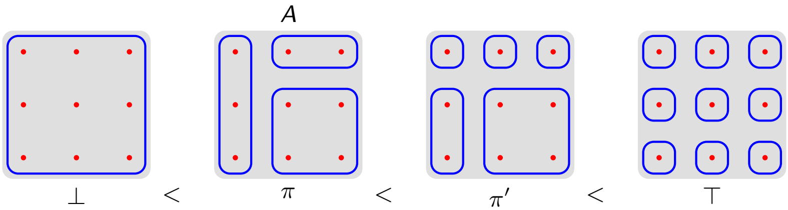

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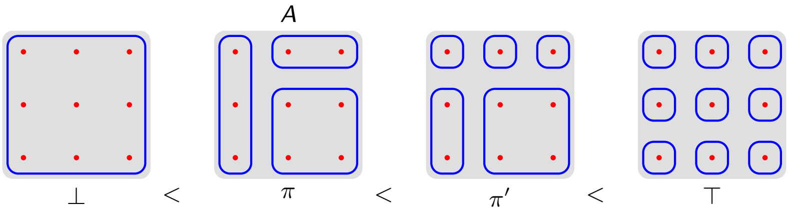

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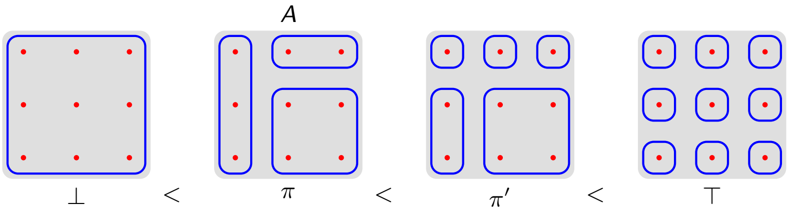

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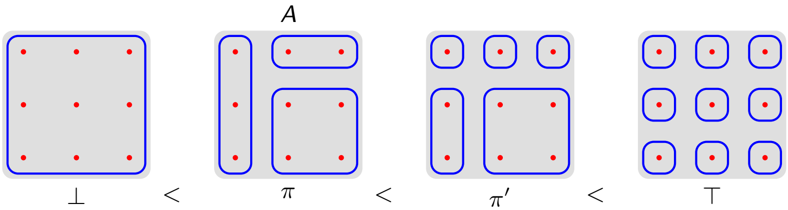

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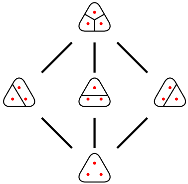
$\mathcal{P}A =$ 

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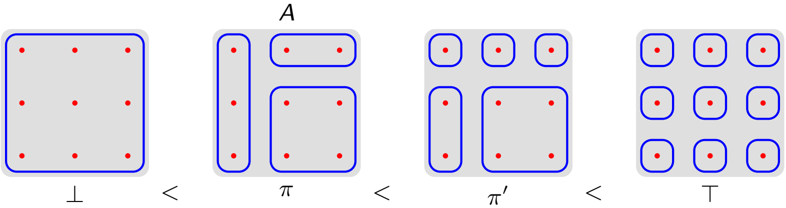

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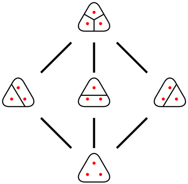
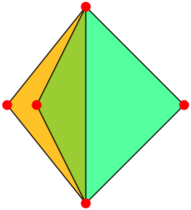
$A =$  $\mathcal{P}A =$  $PA =$ 

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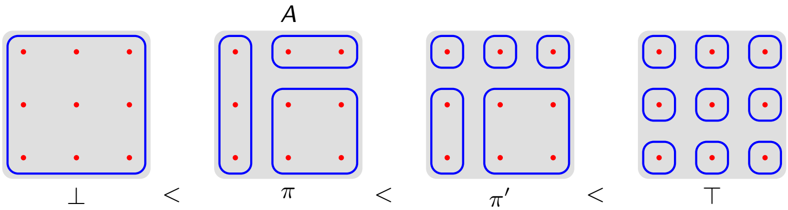

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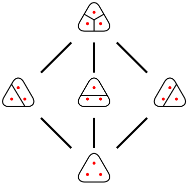
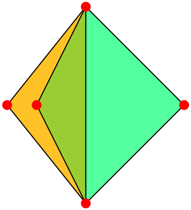
$A =$  $\quad \mathcal{P}A =$  $\quad PA =$ 

$\widehat{P}(A) \simeq S^2 \cup$ (equatorial disc) $\simeq S^2 \vee S^2$.

Products of partitions

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We have not yet understood the structure of this.

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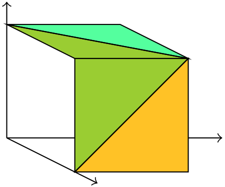

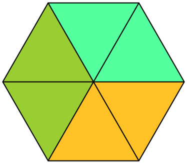

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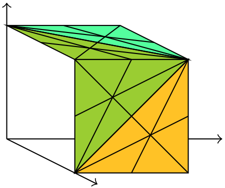

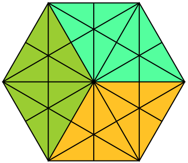

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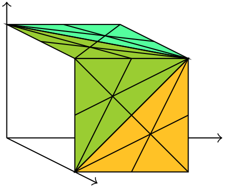

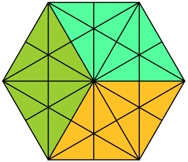

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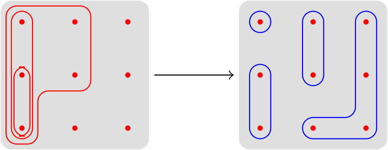

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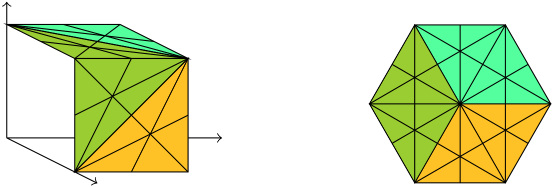

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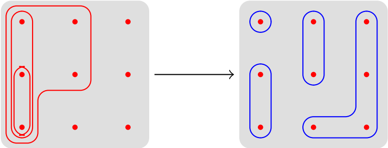

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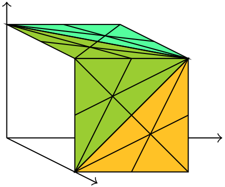

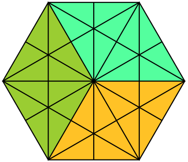

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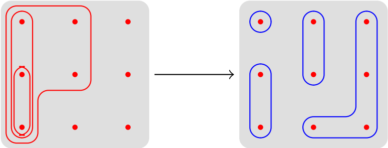

This gives $B(WA) \to P(A)$ and $S^{WA} = B(WA)/\partial B(WA) \to \widehat{P}(A)$.
More generally, we can use the monoid structure on PA to get
$B(WA)^N \to P(A)$ and $S^{NWA} \to \widehat{P}(A)$.

Height functions

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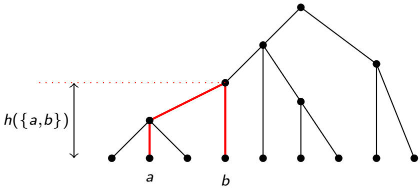

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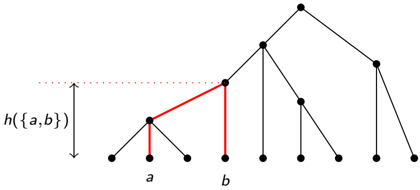

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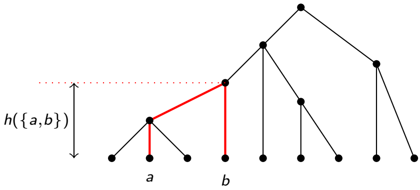

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By a Pontrjagin-Thom construction, we make the spaces $\widehat{P}(n)$ into a based cooperad (a theorem of Ching).

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$\mathrm{Inj}_0(k, \mathbb{R}^n) = \{(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k \mid \sum x_i = 0,\ x_i \neq x_j\} \subseteq W_k \otimes \mathbb{R}^n \subset S^{W_k \otimes \mathbb{R}^n}$.

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These spaces form an operad up to homotopy, as they are homotopy equivalent to the little n-cubes spaces.

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Theorem (Johnson, Arone-Mahowald): $Q(n)$ controls the layers in the Goodwillie tower.

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We can map $\widehat{T}(A)$ to $\widehat{P}(A)$ by sending $B \in \mathcal{T}(A)$ to the corresponding coset partition of A. This respects the actions of $\mathrm{Aff}(A)$ on $\widehat{T}(A)$ and Σ_A on $\widehat{P}(A)$.

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Theorem (Arone-Dwyer):
$(\Sigma^{WA}\widehat{T}(A))_{h\,\mathrm{Aff}(A)} = (\Sigma^{WA}\widehat{P}(A))_{h\Sigma_A} = \overline{\mathrm{SP}}^{p^d}(S^0) = \Sigma^d L(d)$, and so $L(d)$ is the Steinberg summand in $(S^{WA})_{hA}$, which is a Thom spectrum over BA.

Mitchell's complexes

Put $X(A) = (\Sigma^{-d}\mathrm{Bases}(\mathbb{C}[A])_{+} \wedge \widehat{T}(A))_{h\,\mathrm{Aff}(A)}$.

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This was the first known example of a family of finite spectra of type n for all n; an important ingredient of the chromatic theory.

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One can show that $St_*(A)$ has a generator $x_L \in \mathrm{St}_1(A)$ for each $L \leq A$ of order p, subject to relations

$$x_L x_M + x_M x_N + x_N x_L = 0$$

whenever $|L + M + N| < p^3$. The differential is given by $d(x_L) = -1$ for all L.

Chromatic homology

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Let E be Morava E-theory (with formal group G) and put $E_0^\vee L(d) = \pi_0 L_{K(n)}(E \wedge L(d))$. It works out that $E_0^\vee L(*)$ is a contractible DGA over E_0.

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Hopkins-Kuhn-Ravenel introduce the group $\Theta = (\mathbb{Z}/p^\infty)^d$, and a Galois extension E_0' of $\mathbb{Q} \otimes E_0$, with Galois group $\mathrm{Aut}(\Theta)$. For finite groups H, they give a natural isomorphism

$$E_0' \otimes_{E_0} E^0 BH = \mathrm{Map}(\mathrm{Hom}(\Theta^*, H)/H, E_0')$$

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This is closely related to old conjectures of Hopkins, about homological algebra for the ring of E-theory power operations.