Consequences of the Chromatic Splitting Conjecture

Neil Strickland

July 31, 2023

The Chromatic Splitting Conjecture

The CSC (due to Hopkins) is about the structure of $\alpha_n(S) = L_{n-1}L_{K(n)}S$.

Technical note: throughout this talk, S denotes the p-complete sphere spectrum, and we work in the category of S-modules. Symbols like MU refer to the p-completed versions.

We put $S_n^d = L_n S^d$ and $\widehat{S}_n^d = L_{K(n)}S^d$. Given a ring spectrum R and variables z_i of odd degree d_i and chromatic height n_i, we define

$$E_R[z_1, \ldots, z_m] = R \wedge \bigwedge_i (S \vee S_{n_i}^{d_i}) = \bigvee_{I \subseteq \{1,\ldots,m\}} S_{\min_I n_i}^{\sum_I d_i}$$

To expand this out, remember that $S_n^i \wedge S_m^j \simeq S_{\min(n,m)}^{i+j}$.

We introduce variables x_{in} for $0 \leq i < n$ of height i and degree $1 - 2(n-i)$. The CSC says that there are maps $x_{in} \colon S_i^{1-2(n-i)} \to \alpha_n(S)$ inducing

$$E_{S_{n-1}}[x_{0n}, \ldots, x_{n-1,n}] \simeq \alpha_n(S).$$

For example:

$$\begin{aligned}\alpha_3(S) = L_2 L_{K(3)} S &\simeq S_2 \wedge (S \vee S_0^{-5}) \wedge (S \vee S_1^{-3}) \wedge (S \vee S_2^{-1}) \\ &\simeq S_2 \vee S_2^{-1} \vee S_1^{-3} \vee S_1^{-4} \vee S_0^{-5} \vee S_0^{-6} \vee S_0^{-8} \vee S_0^{-9}.\end{aligned}$$

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The conjecture is false in general

- Beaudry has proved that the CSC is false for $n = p = 2$.
- It may still be true when p is large relative to n.
- When p is large the question is in principle purely algebraic, by work starting with Franke, later versions e.g. by Patchkoria-Pstragowski.
- We could also take an ultraproduct over primes, following Barthel-Schlank-Stapleton.
- This talk will investigate a complex set of consequences that would follow from the CSC. These appear to be internally consistent, although there are many ways in which that could fail. This makes the CSC more interesting and more plausible.
- Conjecture: the resulting algebraic and combinatorial patterns are indirectly relevant, even if CSC fails.

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Generators; Localisation relations

The extended Morava stabiliser group Γ_n of height n acts on $W\mathbb{F}_{p^n}$ with

$$H^*(\Gamma_n; \mathbb{Q} \otimes W\mathbb{F}_{p^n}) = E_{\mathbb{Q}_p}[x_{in} \mid 0 \leq i < n] \qquad x_{in} \in H^{2(n-i)-1}.$$

These elements x_{in} should be related via the $K(n)$-based Adams spectral sequence to the elements x_{in} in the CSC.
Also, $x_{n-1,n} \colon S_{n-1}^{-1} \to L_{n-1} L_{K(n)} S$ should come from the known element $\zeta_n \colon S^{-1} \to L_{K(n)} S$ (defined using $\ker(\det \colon \Gamma_n \to \mathbb{Z}_p^\times)$).

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The spherical category

- Fix p and $N \geq 0$, and assume CSC holds for $n \leq N$.
- Let $\mathcal{C}$ be the closure of $\mathcal{C}_1 = \{S_0, S_1, \ldots, S_N, \widehat{S}_1, \ldots, \widehat{S}_N\}$ under (de)suspension and finite coproducts (remembering $\widehat{S}_0 = S_0$).
- Claim: $\mathcal{C}$ is closed under smash products and $F(-,-)$ and $L_{E(n)}$ and $L_{K(n)}$, so is a closed symmetric monoidal category.
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- For $n \geq m$ we have $F(S_n, S_m) = S_m$ and $F(S_n, \widehat{S}_m) = \widehat{S}_m$.
- For $n < m$ both S_n and $\widehat{S}_n$ are $K(m)$-acyclic, so $F(S_n, \widehat{S}_m) = F(\widehat{S}_n, \widehat{S}_m) = 0$.
- For $n < m$, apply $F(S_n, -)$ to the chromatic fracture square for S_m, giving $F(S_n, S_m) = \bigvee_{I \neq \varnothing} F(S_n, S^{\bullet-1}_{\min(I, m-1)})$; then repeat recursively.
- Similar methods give $F(\widehat{S}_n, S_m)$ and $F(\widehat{S}_n, \widehat{S}_m)$.
- Recall X is strongly dualisable iff $F(X, S_N) \wedge X \to F(X, X)$ is iso. This only holds if X is a wedge of copies of $S_N^{\bullet}$.
- Put $R = \pi_0(\mathcal{C}) = \mathbb{N}[s^{\pm 1}]\{S_0, \ldots, S_N, \widehat{S}_1, \ldots, \widehat{S}_N\}$.
- By spelling out the combinatorics, we get a product on R and a map $F \colon R \times R \to R$.
- Fact: the product is commutative and associative and satisfies $F(x, F(y, z)) = F(xy, z)$, as predicted by CSC.
- This is all about isomorphism classes; what about morphisms?

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$$\bigvee_I S^{\bullet-1}_{min(I,n-1)} \xrightarrow{u} S_n \xrightarrow{\left[\begin{smallmatrix} i \\ -j \end{smallmatrix}\right]} S_{n-1} \vee \widehat{S}_n \xrightarrow{[\,\eta\ \ v\,]} \bigvee_I S^{\bullet}_{min(I,n-1)}$$

  The maps u and v have components $u_I\colon S^{\bullet}_{\bullet} \to S_n$ and $v_I\colon \widehat{S}_n \to S^{\bullet}_{\bullet}$

- Additional conjecture: any composite $S_m \xrightarrow{i} S^{\bullet}_{\bullet} \xrightarrow{u_I} S_n$ with $m \geq n$ is zero. This is true but not obvious when $N = 1$.
- Assuming this: we hope to determine the composition maps $F(Y,Z) \wedge F(X,Y) \to F(X,Z)$.
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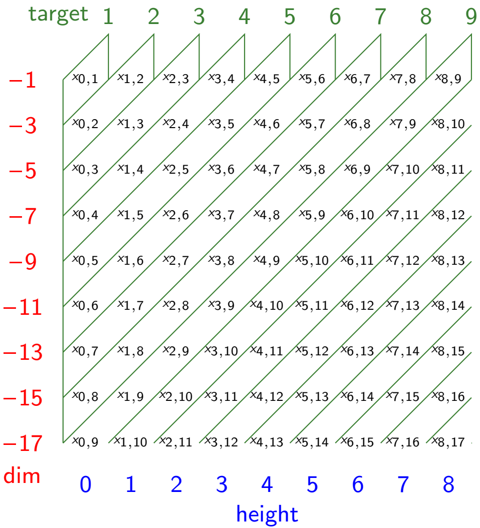
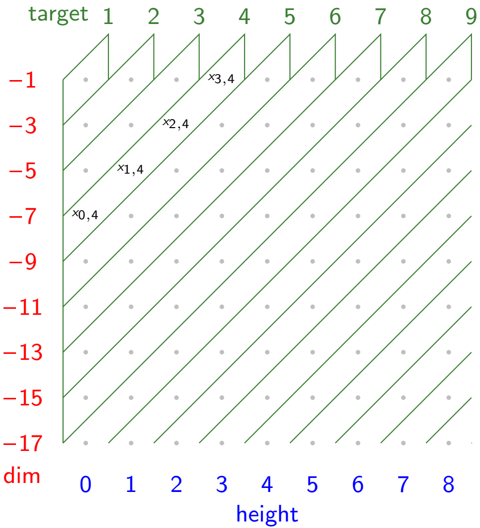


$\alpha_n(S) = E_{S_{n-1}}[x_{0n}, \ldots, x_{n-1,n}]$

x_{in} has height i, target n

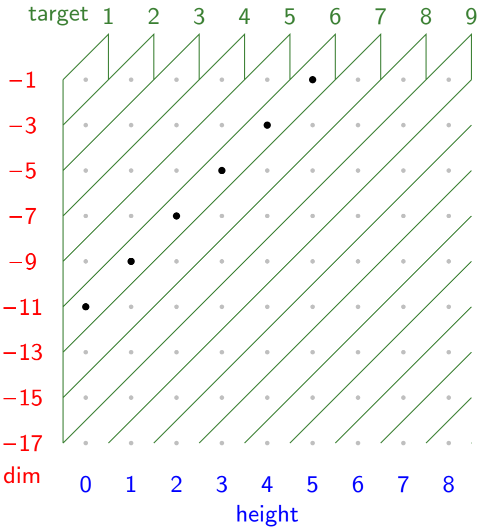
and dimension $1 - 2(n - i)$

Charts



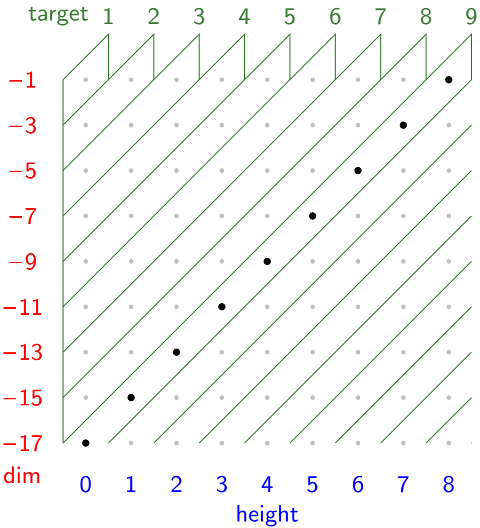
$$\alpha_4(S) = E_{S_3}[x_{04}, x_{14}, x_{24}, x_{34}]$$

Charts



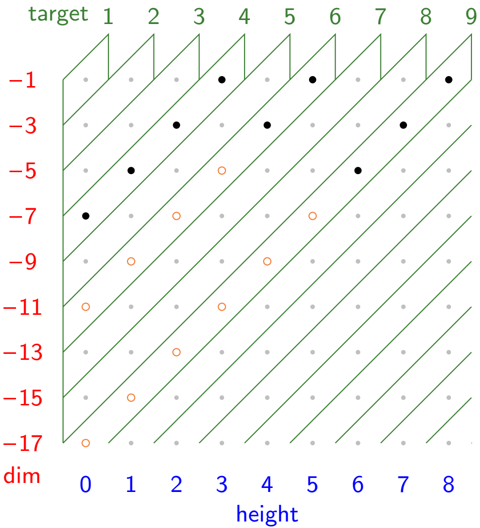
$$\alpha_6(S) = E_{S_5}[x_{06}, \ldots, x_{56}]$$

Charts



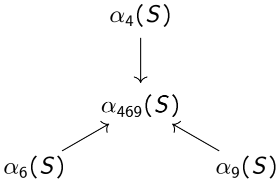
$\alpha_9(S) = E_{S_8}[x_{09}, \ldots, x_{89}]$

Charts



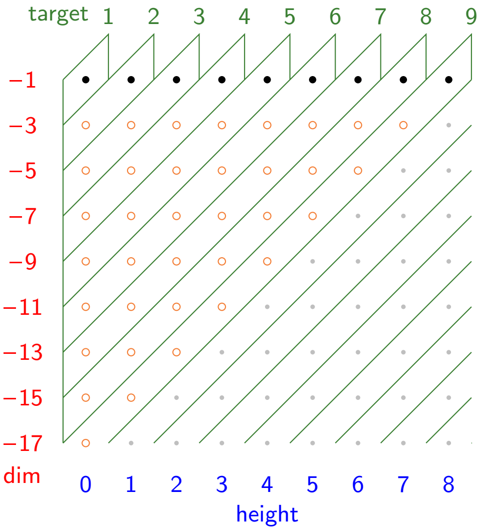
Put $\alpha_{469} = \alpha_4 \circ \alpha_6 \circ \alpha_9$

$$= L_3 L_{K(4)} L_{K(6)} L_{K(9)}$$



$\alpha_{469}(S)$ is exterior over S_3 on 9 generators indicated in black. Circles are shadowed generators: present but equal to zero.

Charts

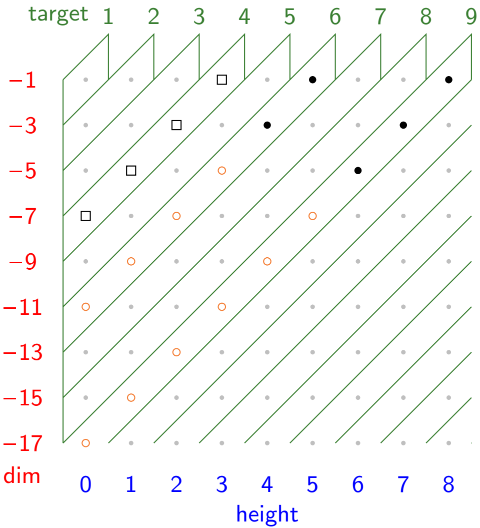


$\alpha_{1\dots9} = L_{K(0)} L_{K(1)} \cdots L_{K(9)}$

$\alpha_{1\dots9}(S)$ is exterior over S_0 on

$x_{01}, x_{12}, \dots, x_{89}$ (all degree -1)

Charts



Put $\phi_{469} = L_{K(4)} \circ \alpha_6 \circ \alpha_9$

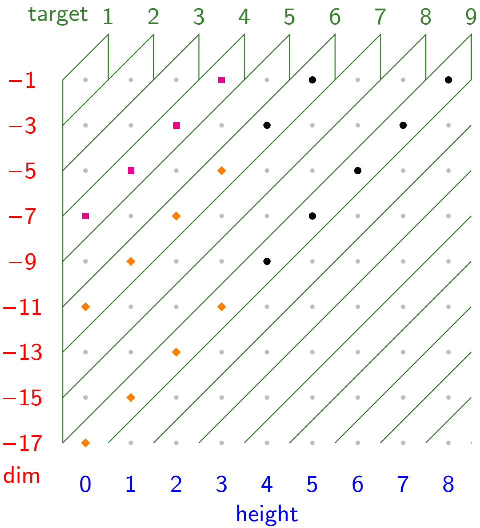
$$= L_{K(4)} L_{K(6)} L_{K(9)}$$

$$\alpha_{69}(S) \to \phi_{469}(S) \leftarrow \widehat{S}_4$$

$\phi_{469}(S)$ is exterior over $\widehat{S}_4$ on 5 generators marked in black.
Circles are shadowed generators: present but equal to zero.

All summands in this exterior algebra are just $\widehat{S}_4^d$.

Charts



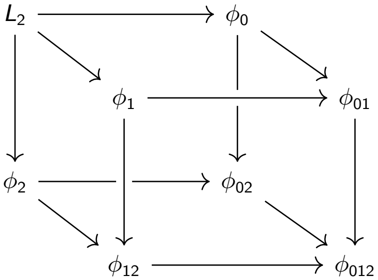
$\widehat{S}_4 \wedge \widehat{S}_6 \wedge \widehat{S}_9$ is a wedge of terms indexed by admissible monomials in the indicated generators

If only • present: term is $\widehat{S}_4$

If any more present: at least one must be ◆, and the term is S_i for some $i < 4$.

Chromatic fracture

The following cube of functors is homotopy cartesian (where $\phi_{02} = L_{K(0)} L_{K(2)}$ etc.):



Homotopy cartesian means:

- L_2 maps by an equivalence to the holim of the rest of the diagram; or
- The total fibre of the cube is zero.

Rules for total fibres:

- tfib(cube) = fib(tfib(face) → tfib(opposite face))
- tfib(square) = fib(fib(edge) → fib(opposite edge))

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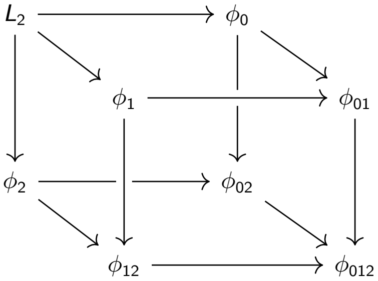

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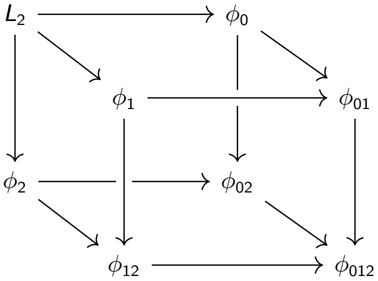

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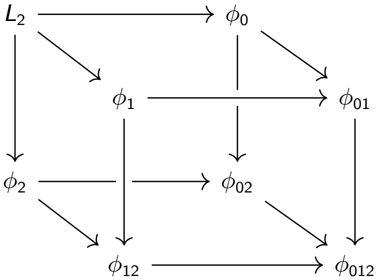

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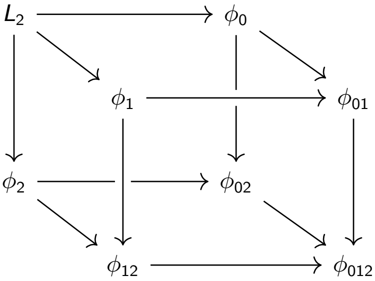

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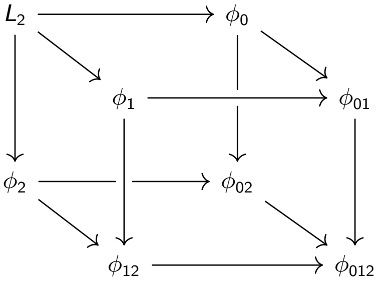

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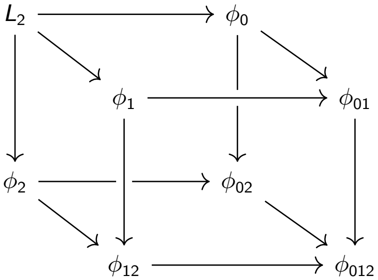

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Aside on spectral sequences

The chromatic fracture cube gives a spectral sequence

$$E^1_{pq} = \prod_{|A|=p} \pi_q(\phi_A(X)) \Longrightarrow 0,$$

where A runs over subsets of $\{0,1,2\}$ and $\phi_\varnothing = L_2$.

For a formally similar situation, take a space $X = U_0 \cup U_1 \cup U_2$, and put $U_{02} = U_0 \cap U_2$ etc. There is a Mayer-Vietoris spectral sequence

$$E_0^{pq} = \prod_{|A|=p} C^q(U_A), \quad E_1^{pq} = \prod_{|A|=p} H^q(U_A) \Longrightarrow 0.$$

Consider the exterior algebra $E = E[e_0, e_1, e_2]$ with basis $\{e_A \mid A \subseteq \{0,1,2\}\}$.
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Spectral sequence of this type deserve further study.

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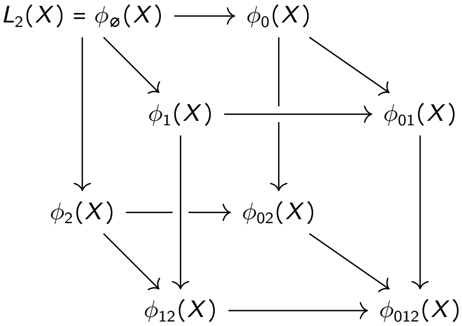

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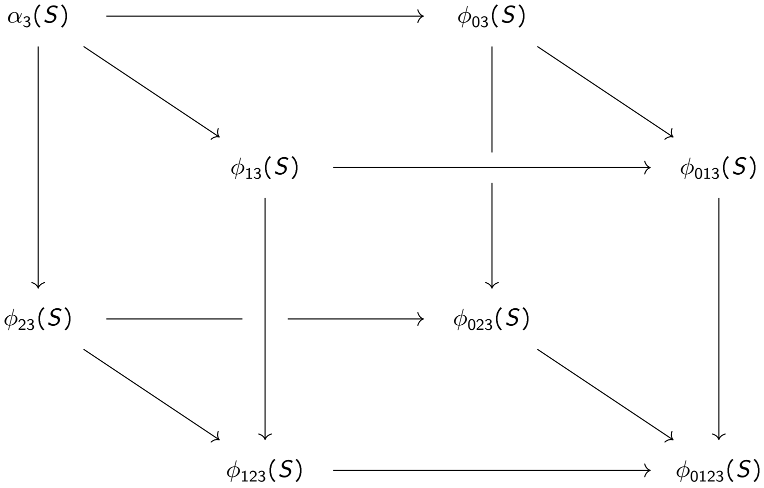
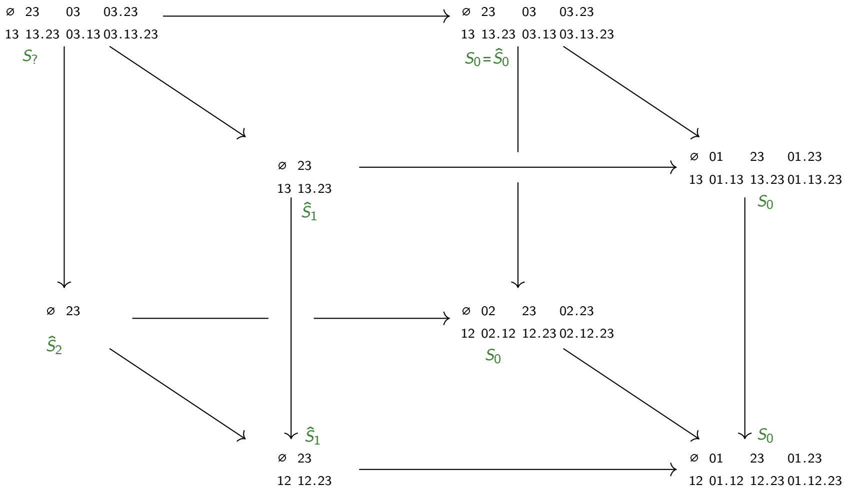
Chromatic splitting and chromatic fracture



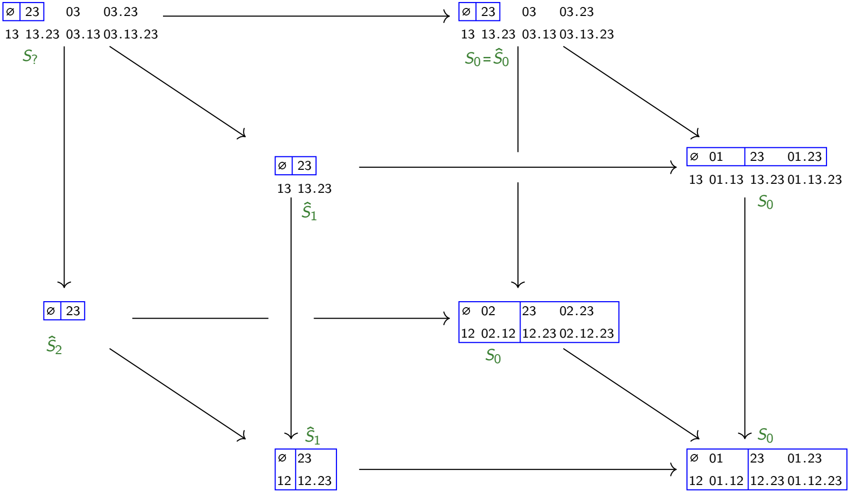
Apply the fracture cube to $\widehat{S}_3$ to get a homotopy cartesian cube as above.
Is this consistent with the Chromatic Splitting Conjecture?

Chromatic splitting and chromatic fracture



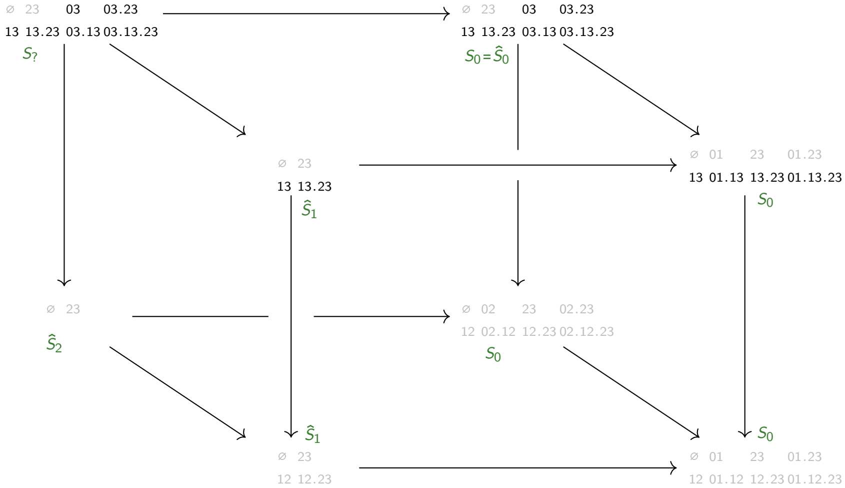
Notation: e.g. 01.13 = $x_{01}x_{13}$; also ∅ = 1.
This diagram should be homotopy cartesian.

Chromatic splitting and chromatic fracture



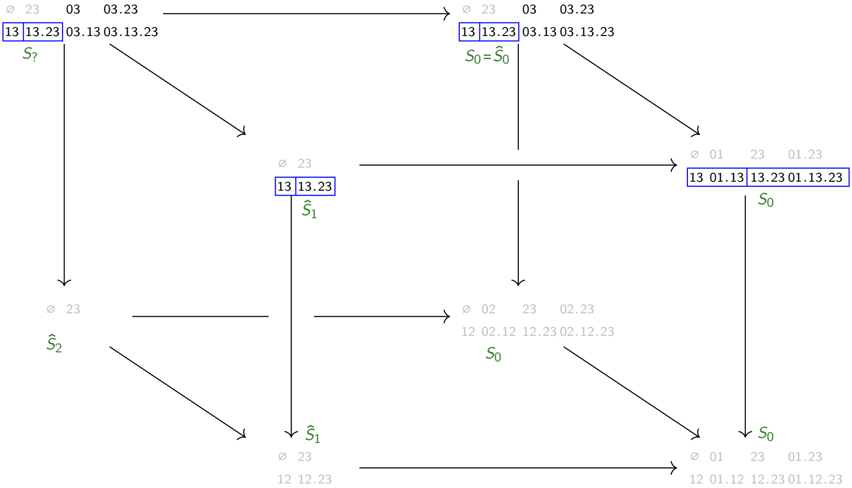
This subdiagram consists of two copies of the fracture cube for S_2 and so is homotopy cartesian.

Chromatic splitting and chromatic fracture



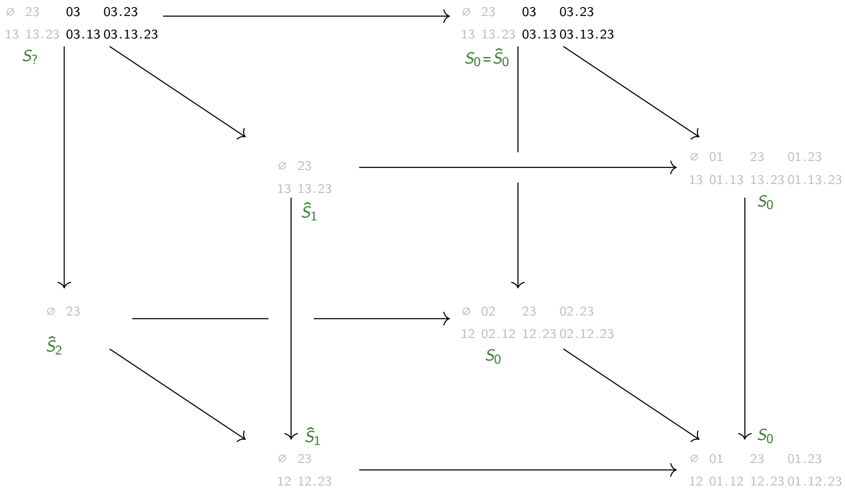
We can remove that subdiagram without changing the total fibre.

Chromatic splitting and chromatic fracture



This subdiagram consists of two copies of the fracture square for S_1 and so is homotopy cartesian.

Chromatic splitting and chromatic fracture



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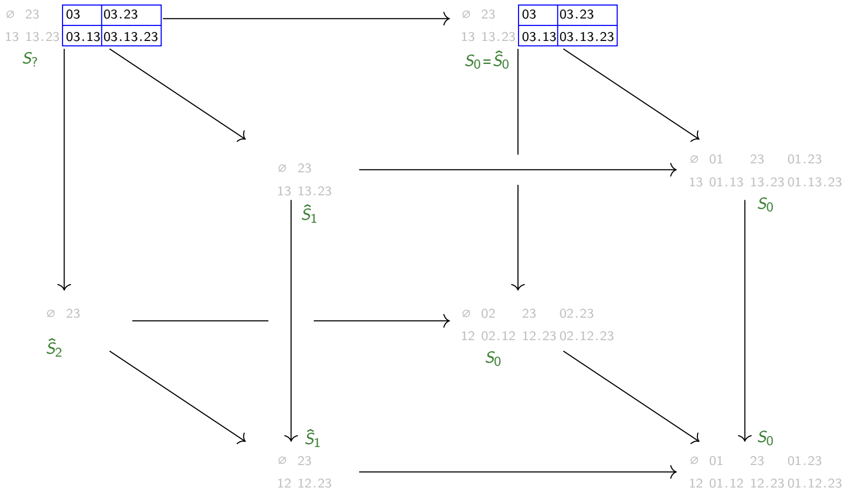
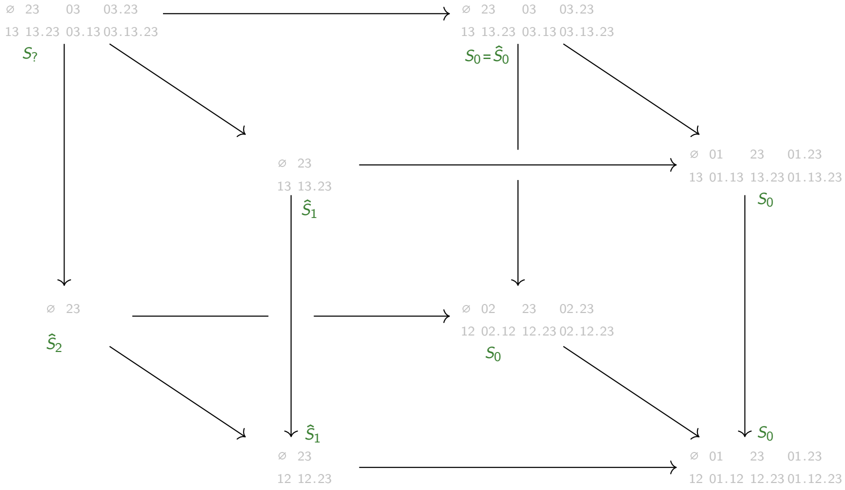


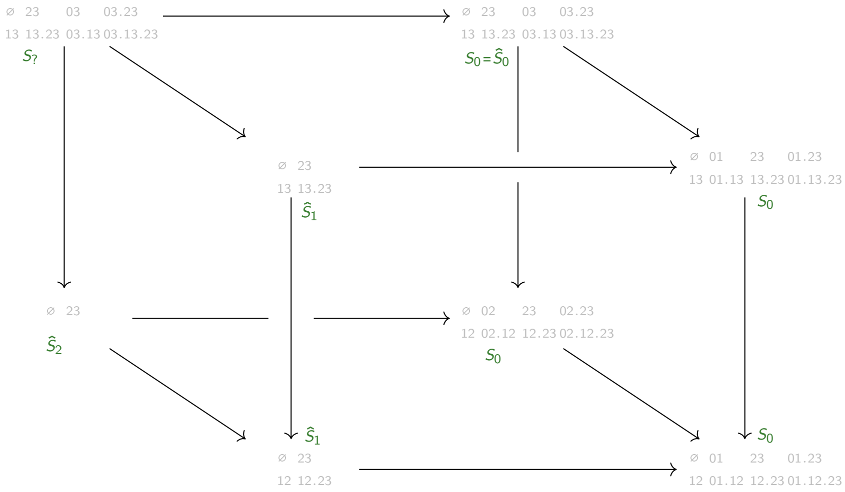
This subdiagram consists of four copies of the fracture interval for S_0 and so is homotopy cartesian.

Chromatic splitting and chromatic fracture



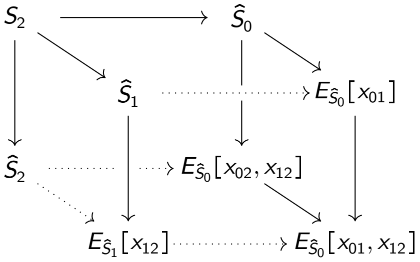
After removing that subdiagram we see that the original diagram was homotopy cartesian, as required.

Chromatic splitting and chromatic fracture



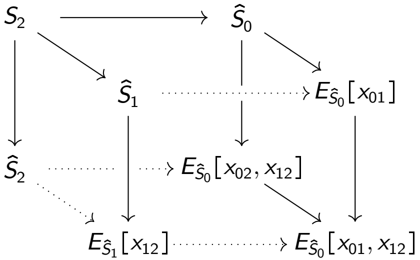
Similarly, CSC implies that the chromatic fracture hypercube for $\alpha_A(S) = L_{n-1}(\phi_A(S))$ is a sum of the hypercubes for various S^d_m.

Chromatic splitting and chromatic fracture



- According to CSC we should have a homotopy cartesian cube as above.
- Dotted arrows are defined using CSC. Solid arrows exist unconditionally.
- Everything but S_2 has a decreasing filtration by powers of the ideal generated by all x_{in}. There is a compatible filtration of S_2.
- $\mathrm{gr}_0(S_2) = \widehat{S}_2$; $\mathrm{gr}_1(S_2) = \widehat{S}_0^{-4} \vee \widehat{S}_1^{-2}$; $\mathrm{gr}_2(S_2) = \widehat{S}_0^{-5} \vee \widehat{S}_0^{-4}$
- In general, the CSC implies that S_n has a finite decreasing filtration where the associated graded is a wedge of $K(m)$-local spheres which can be described combinatorially. Multiplicative properties are unclear.

Chromatic splitting and chromatic fracture



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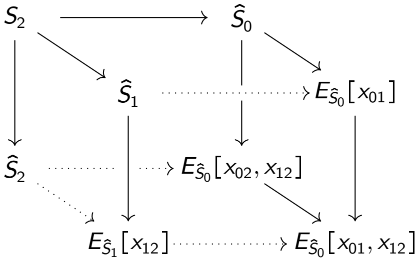

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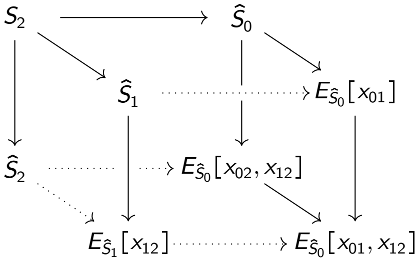

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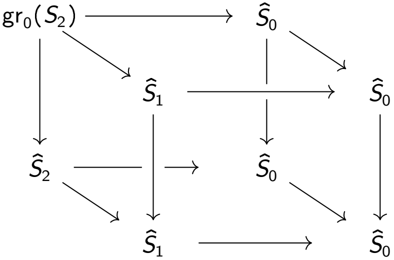

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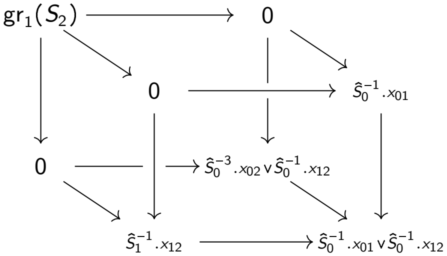

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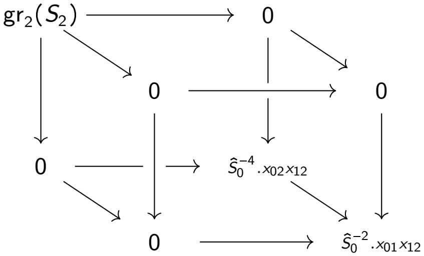

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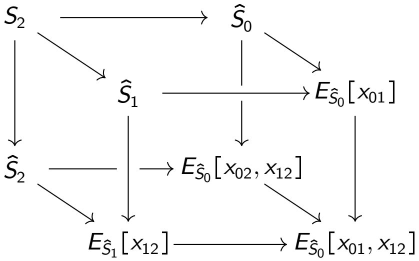

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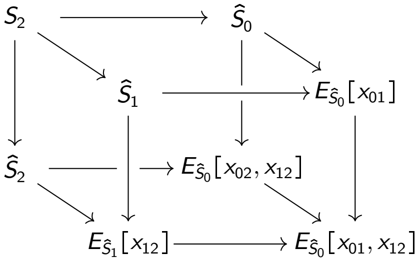

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The associated graded object $\mathrm{gr}_*(S_n)$ is conjecturally as follows:

- For any sequence $u = (u_0 < u_1 < \cdots < u_r = n)$ we have $z_u\colon \widehat{S}_{u_0}^{2(u_0-n)} \to \mathrm{gr}_r(S_n)$.
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- Put $\chi_n(X) = \dim_{K(n)_*}(K(n)_{\text{even}}(X)) - \dim_{K(n)_*}(K(n)_{\text{odd}}(X))$ (assuming that the dimensions are finite).
- For the X that we have considered: $\chi_n(X)$ is probably 0, occasionally 1.
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Further questions

- For $U \subseteq P\{0, \ldots, N\}$ closed upwards, put $\theta_U(X) = \underleftarrow{\text{holim}}_{A \in U} \phi_A(X)$.
- In work with Bellumat we showed that this class of functors contains L_n and $L_{K(n)}$ and is closed under composition and certain homotopy limits.
- We believe that CSC implies a splitting of all $\theta_U(S)$, but have not completed this analysis.
- Ravenel has defined ring spectra
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- The CSC is about $\phi_A(T(0))$ and $\alpha_A(T(0))$.
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- This is also relevant for the Telescope Conjecture.

Further questions

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