Stable Homotopy Theory

or

Topology with negative spheres.

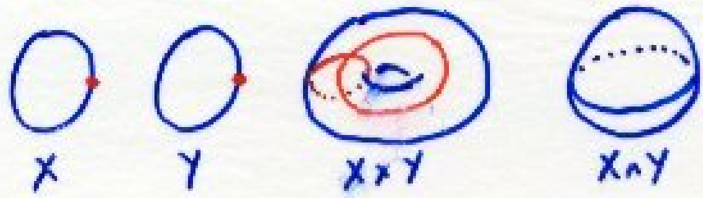
Unstable homotopy

- Work with pointed spaces X with basepoint $o_X = *_X \in X$.
- Eg $S^n = \mathbb{R}^n \cup \{\infty\}$ with basepoint ∞ $(S^n \simeq \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\})$.
- $\tilde{H}_* X$ is the reduced homology of X eg $\tilde{H}_* S^n = \mathbb{Z}$ in degree n.
- $X \vee Y := (X \amalg Y)/(*_X \sim *_Y)$

  $\tilde{H}_*(X \vee Y) = \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$

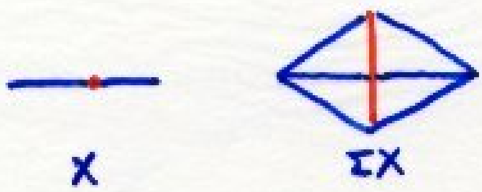
- $X \wedge Y := (X \times Y)/(X \vee Y)$

  $S^n \wedge S^m = S^{n+m}$



- $\tilde{H}_*(X \wedge Y) \sim \tilde{H}_*(X) \otimes \tilde{H}_*(Y)$
- $\Sigma^n X = S^n \wedge X$

  $\Sigma^n S^m = S^{m+n}$; $\Sigma^n B^m = B^{m+n}$



- $\tilde{H}_* \Sigma^n X = \tilde{H}_{*-n} X$
- $F(X, Y) = \{ \text{continuous } f : X \to Y \mid f(*_X) = *_Y \}$

  $[X, Y] = \pi_0 F(X, Y) = \{$ homotopy classes of maps $X \to Y\}$.
- $\pi_n X := [S^n, X]$. This is a group when $n > 0$ and is abelian when $n > 1$.

Cofibres

- Given $f : X \to Y$ we put $Cf = Y \cup_X (X \wedge I)$

- If f is the inclusion of a subspace then $Cf \to Y/X$ is a homotopy equivalence.

- There is a long exact sequence

$$\cdots \to \tilde{H}_* X \to \tilde{H}_* Y \to \tilde{H}_* Cf \to \tilde{H}_* \Sigma X \to \tilde{H}_* \Sigma Y \to \cdots$$

so $\quad \tilde{H}_* Cf = \operatorname{coker}(\tilde{H}_* f)$ "$\oplus$" $\ker(\tilde{H}_* \Sigma f)$

- There is a long exact sequence

$$\begin{array}{lr}
[X, Z] \leftarrow [Y, Z] \leftarrow [Cf, Z] \leftarrow & \text{(based sets)} \\
{}[\Sigma X, Z] \leftarrow [\Sigma Y, Z] \leftarrow [\Sigma Cf, Z] \leftarrow & \text{(groups)} \\
{}[\Sigma^2 X, Z] \leftarrow [\Sigma^2 Y, Z] \leftarrow [\Sigma^2 Cf, Z] \leftarrow \cdots & \text{(abelian groups)}
\end{array}$$

Why negative spheres?

- We'll construct a category $\mathcal{F}$ of finite spectra by "adjoining negative spheres" to the homotopy category of finite based simplicial complexes.

- The proof that $\pi_n(X)$ is an abelian group involves S^{n-2}. Unstably it only works for $n \geqslant 2$, but in $\mathcal{F}$ it holds for all n.

- Theorem (Freudenthal): $\pi_{k+n}(\Sigma^n Y)$ & $[\Sigma^n X, \Sigma^n Y]$ are independent of n for $n \gg 0$.

  In $\mathcal{F}$, we have independence for all n.

- In $\mathcal{F}$, the first nontrivial homotopy group of X is the same as the first nontrivial homology group (the Hurewicz theorem). Unstably there are awkward modifications for low degrees.

- Given a space X one can find N and Y such that $\tilde{H}_k Y \simeq \tilde{H}^{N-k} X$. (This is called Spanier-Whitehead duality).

  In $\mathcal{F}$ we can have $N=0$ & $Y =: DX$ is canonical.

- $MO_n :=$ { cobordism classes of n-manifolds } $\quad MO_* = \bigoplus_n MO_n$

  Theorem (Thom): $\exists$ spaces $MO(k)$ with $MO_n = \pi_{n+k} MO(k)$ for $k \gg 0$ (and $MO_* = (\mathbb{Z}/2)[x_2, x_4, x_5, x_6, x_8, \ldots]$)

  There is a spectrum MO with $\pi_n MO = MO_n$.

- Bott periodicity says $\pi_*(BO)$ is 8-periodic in positive degrees. The spectrum KO has $\pi_*(KO)$ periodic in all degrees.

The Spanier-Whitehead Category

- A finite spectrum is just a pair (n, X) (to be thought of as $\Sigma^n X$) where $n \in \mathbb{Z}$ and X is a finite based complex.

- We define $[(n, X), (m, Y)] := [\Sigma^{N+n} X, \Sigma^{N+m} Y]$ for $N \gg 0$

- These are the morphism sets for the category $\mathcal{F}$ of finite spectra, also called the Spanier-Whitehead category.

- A more complex procedure expands $\mathcal{F}$ to give Boardman's category $\mathcal{B}$ of (possibly infinite) spectra.

- Given $P, Q \in \mathcal{B}$ we can define $P \vee Q$, $P \wedge Q$, ΣP, $\Sigma^{-1} P$, $H_* P$ and $\pi_* P$. Properties are as for spaces except that $[\Sigma P, \Sigma Q] = [P, Q]$ and $\pi_* \Sigma P = \pi_{*-1} P$.

- Given $f: P \to Q$ in $\mathcal{B}$ we can define $Cf \in \mathcal{B}$ and get long exact sequences

$$\cdots \to H_* \Sigma^{-1} Cf \to H_* P \to H_* Q \to H_* Cf \to H_* \Sigma P \to \cdots$$

$$\cdots \to [T, \Sigma^{-1} Cf] \to [T, P] \to [T, Q] \to [T, Cf] \to [T, \Sigma P] \to \cdots$$

$$\cdots \leftarrow [\Sigma^{-1} Cf, T] \leftarrow [P, T] \leftarrow [Q, T] \leftarrow [Cf, T] \leftarrow [\Sigma P, T] \leftarrow \cdots$$

- There is a duality functor $D: \mathcal{F}^{op} \to \mathcal{F}$ with $D^2 X = X$ and $[U \wedge V, W] \simeq [U, DV \wedge W]$. and $H_* X = H^{-*} DX$.

If $X \subset S^N$ and $Y = S^N \setminus X$ then $D(n, X) = (1-n-N, Y)$.

Computation

- $[P,Q]$ is recursively computable from simplicial description of P and Q (by impractical algorithms).

- In practice, $[P,Q]$ can often be determined by hard calculations with the Adams spectral sequence

- $$\pi_n S^0 = \begin{cases} 0 & \text{for } n<0 \\ \mathbb{Z} & \text{for } n=0 \\ \text{finite} & \text{for } n>0 \end{cases}$$

- $\pi_* S^0$ is a ring, in which all elements in degree >0 are nilpotent (by Nishida's theorem).

- $\pi_0 S^0 = \mathbb{Z}$
  $\pi_1 S^0 = (\mathbb{Z}/2)\{\eta\}$
  $\pi_2 S^0 = (\mathbb{Z}/2)\{\eta^2\}$
  $\pi_3 S^0 = (\mathbb{Z}/24)\{\nu\}$
  $\pi_4 S^0 = 0$
  $\pi_5 S^0 = 0$
  $\pi_6 S^0 = (\mathbb{Z}/4)\{\nu^2\}$

- $[P,Q]\otimes\mathbb{Q} \simeq \mathrm{Hom}\,(H_*(P;\mathbb{Q}),\, H_*(Q;\mathbb{Q}))$

Thom spectra & duality

- Given a vector bundle V over a finite complex X, the Thom space X^V is just the one-point compactification of the total space of V.

- If V is a constant bundle of dimension n then $X^V = \Sigma^n X_+$.

- $\tilde{H}^*(X^V)$ is a free module over H^*X on one generator, in degree equal to the dimension of V. (Thom isomorphism)

- Given another bundle W we define a Thom spectrum X^{V-W} as $(-N, X^{V \oplus U}) \in \mathcal{F}$, where U is chosen such that $V \oplus U$ is isomorphic to the N-dimensional constant bundle.

- For a compact closed manifold M with tangent bundle T we have $D(M^{V-W}) = M^{W-V-T}$. (Atiyah duality)

Poincaré duality follows easily from this.

Stable splitting

- There are many spaces X whose corresponding spectra split up in unexpected and interesting ways.
- Put $G_{k,n} = \{ V \leq \mathbb{C}^n \mid \dim(V) = k \}$ and let A be the bundle whose fibre at V is the space of antihermitian endomorphisms of V.

  Then $U(n) \simeq \bigvee_{k=0}^{n} G_{k,n}^{A}$ in $\mathcal{F}$. (Haynes Miller.)
- The space $\Omega S^{n+1} = F(S^1, S^{n+1})$ splits in $\mathcal{B}$ as $\bigvee_{k \geq 0} S^{nk}$.
- The space $\Omega^2 S^3 = F(S^2, S^3)$ splits in $\mathcal{B}$ as $\bigvee_{k \geq 0} (F_k \mathbb{R}^2)^{V_k}$

  where $F_k \mathbb{R}^2 = \{ A \subseteq \mathbb{R}^2 \mid |A| = k \}$ and the fibre of V_k at A is $Map(A, \mathbb{R})$.
- Suppose that a finite group G acts on X, that $H_* X$ is a p-torsion group, and that $e \in \mathbb{F}_p[G]$ satisfies $e^2 = e$.

  Then $X = eX \vee (1-e)X$, where $H_*(eX; \mathbb{F}_p) = e \cdot H_*(X; \mathbb{F}_p)$
- Eg $G = GL_n(\mathbb{F}_p)$, e = Steinberg idempotent, X = any of various spaces associated functorially to vector spaces over $\mathbb{F}_p$.
- Eg $G = \Sigma_n$, $X = Y^{(n)} = Y \wedge \dots \wedge Y$ with evident G-action. Splitting of X gives important examples for chromatic theory.
- There is a Krull-Schmidt theorem saying that p-torsion finite spectra can be split uniquely into indecomposables.

Cobordism

- Given a space X, a geometric n-chain in X is a pair (M, f) where M is a compact smooth n-manifold (possibly with boundary) and $f : M \to X$.

- The set $GC_n(X)$ of isomorphism classes of geometric n-chains is a commutative monoid under disjoint union.

- There is a boundary map $\partial : GC_n(X) \to GC_{n-1}(X)$ given by $\partial[M, f] = [\partial M, f|_{\partial M}]$.

- This satisfies $\partial^2 = 0$, so we can define $MO_*(X) = \ker(\partial)/\mathrm{im}(\partial)$

- Theorem (Thom): $MO_*(X) = \pi_*(MO \wedge X_+) = MO_*(\text{point}) \otimes_{\mathbb{F}_2} H_*(X; \mathbb{F}_2)$.

- Repeat with oriented manifolds to get $MSO_*(X) = \pi_*(MSO \wedge X_+)$. This is not determined by H_*X.

- Repeat with (stably almost) complex manifolds to get $MU_*(X) = \pi_*(MU \wedge X_+)$.

- $MU_*(\text{point})$ is unnaturally isomorphic to $\mathbb{Z}[x_1, x_2, x_3, \dots]$ and is canonically identified with Lazard's universal ring for formal groups. This makes an important link with formal group theory.

- The Adams-Novikov spectral sequence gives a method for computing $\pi_* S^0$ starting with the cohomology of the stack of formal groups.

Morava K-theory

- $MU_* X$ is a very powerful invariant of X, but it is also very large & sometimes hard to compute.

- The Morava K-theories are spectra $K(p,n)$ defined for p prime and $0 \leq n \leq \infty$. Together they are almost as powerful as MU but much easier to work with.

- $K(p,0)_* X = H_*(X;\mathbb{Q})$
  $K(p,\infty)_* X = H_*(X;\mathbb{F}_p)$
  $K(p,n)_*(\text{point}) = \mathbb{F}_p[v_n, v_n^{-1}] \qquad |v_n| = 2p^n-2$.

- If X is finite then $d(p,n,X) := \dim_{\mathbb{F}_p[v_n,v_n^{-1}]} K(p,n)_* X$ is finite

  and $d(p,n,X) \leq d(p,n+1,X)$. Moreover,

  $K(p,n)_* X = H_*(X;\mathbb{F}_p)[v_n, v_n^{-1}]$ for $n \gg 0$.

- If $E = K(p,n)$ then $E_*(X \wedge Y) = E_* X \otimes_{E_*(pt)} E_* Y$ for all X, Y.

  Any other theory E with this property is determined by the $K(p,n)$'s.

- We also have $K(p,n)^* X = \mathrm{Hom}_{\mathbb{F}_p[v_n,v_n^{-1}]}(K(p,n)_* X, \mathbb{F}_p[v_n,v_n^{-1}]) = K(p,n)_* DX$.

- If G is a finite group then $K(p,n)^* BG$ is a Frobenius algebra over $\mathbb{F}_p[v_n,v_n^{-1}]$. It is known explicitly in many cases.

- If $G = C_{p^{k_1}} \times \dots \times C_{p^{k_r}}$ then $K(p,n)^* BG = \mathbb{F}_p[v_n^{\pm 1}, x_1, \dots, x_r]/(x_1^{p^{nk_1}}, \dots, x_r^{p^{nk_r}})$

- $K(p,n)_* X$ has a natural action of a certain p-adic analytic Lie group of dimension n^2, the Morava stabiliser group.

Nilpotence theory

- Theorem (Hopkins, Devinatz, Smith):
  If $f : \Sigma^d X \to X$ satisfies $K(p,n)_* f = 0$ for all (p,n)
  then $f^k = 0 : \Sigma^{kd} X \to X$ for $k \gg 0$. (Here $X \in \mathcal{F}$).

- There are similar statements for various other kinds of nilpotence.

- ∘ Anything in $\mathcal{F}$ can be built from spheres by cofibrations & retracts.
  ∘ Often useful to start from something other than spheres & see what can be built. For example, if $H_* X$ is p-torsion then X can be built from $\{ S^n/p \mid n \in \mathbb{Z} \}$ where S^n/p is the cofiber of p times the identity map on S^n.
  ∘ If $f : \Sigma^d X \to X$ is zero then $Cf = X \vee \Sigma^{d+1} X$ $\therefore$ X is a retract of Cf
  ∘ If $f : \Sigma^d X \to X$ is merely nilpotent then X can be built from Cf.
  Thus, the nilpotence theorem helps with building problems.

- $\mathrm{type}_p(X) := \min\{ n \mid K(p,n)_* X \neq 0 \} \in \mathbb{N} \cup \{\infty\}$

- Theorem (Hopkins, Smith): X can be built from Y iff $\mathrm{type}_p(X) \geq \mathrm{type}_p(Y)$ for all p.

- $\mathcal{F}$ is analogous to the category $\mathcal{P}_R$ of perfect complexes (= finite chain complexes of finitely generated projective modules) over a ring R. The space $\mathrm{spec}(R)$ can be recovered from $\mathcal{P}_R$.

  We can apply the recovery procedure to $\mathcal{F}$ to obtain a space $\mathrm{spec}(\mathcal{F})$. The above theorem says that $\mathrm{spec}(\mathcal{F})$ bijects with the set of Morava K-theories.

Eilenberg - MacLane spectra

- A spectrum $X \in \mathcal{B}$ is an Eilenberg-MacLane spectrum if $\pi_n X = 0$ for all $n \neq 0$.
- We write $\mathcal{E}$ for the full subcategory of $\mathcal{B}$ consisting of Eilenberg-MacLane spectra.
- Theorem: the functor $\pi_0 : \mathcal{E} \to \mathcal{A}b$ is an equivalence.

  Thus, given an abelian group A, there is an essentially unique Eilenberg-MacLane spectrum HA with $\pi_0 HA = A$.
- Theorem: $[X, \Sigma^n HA] \simeq H^n(X; A)$ and $\pi_n(HA \wedge X) = H_n(X; A)$.
- $HA = \varinjlim_n \Sigma^{-n}(A \otimes \mathbb{Z}\{S^n\})$ where

  $\mathbb{Z}\{X\} := (\text{free abelian group on } X) / (\text{free abelian group on } \{*_X\})$
- The graded ring $\mathcal{A}^* = [H\mathbb{F}_p, \Sigma^* H\mathbb{F}_p]$ is called the Steenrod algebra. It is complicated but well-understood.
- The Adams spectral sequence gives a way to compute $(\pi_* S^0)^\wedge_p$ starting from $\mathrm{Ext}_{\mathcal{A}^*}(\mathbb{F}_p, \mathbb{F}_p)$.

The Segal conjecture

- Let G be a finite group. Put

  $A(G) = \{ [X]-[Y] \mid X \,\&\, Y \text{ are finite } G\text{-sets} \}$

- This is a ring with $[X]+[X'] = [X \amalg X']$ and

  $[X][X'] = [X \times X']$. There is a ring map $\varepsilon : A(G) \to \mathbb{Z}$.

  with $\ker(\varepsilon) = I$ say. Put $\hat{A}(G) = \varprojlim_n A(G)/I^n$.

- Theorem (conjecture of Segal, proved by Carlsson)

  $\hat{A}(G) \simeq [(0, BG), (0, BG)]$.

- More generally:
  $A(G,H) \sim (G \times H)$-sets on which G acts freely, but
  H need not. We again put $\hat{A}(G,H) =$ completion wrt I_G.

- Theorem (Lewis - May - Steinberger): $\hat{A}(G,H) \simeq [\Sigma^\infty BG_+, \Sigma^\infty BH_+]$

K-theory of categories

- Given any category $\mathcal{C}$ we define a space $B\mathcal{C}$ with a vertex for each object, an edge for each morphism, and an n-simplex for each composable n-tuple of morphisms.

- Suppose we have a functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ making $\mathcal{C}$ a symmetric monoidal category. This induces a map $B\mathcal{C} \times B\mathcal{C} \to B\mathcal{C}$ making $B\mathcal{C}$ a commutative monoid up to homotopy.

- There is a canonical way to convert this to $B\mathcal{C}^+$ which is a group up to homotopy.

- There is a canonical spectrum $K(\mathcal{C})$ such that the stable homotopy groups $\pi_* K(\mathcal{C})$ are the unstable groups $\pi_* B\mathcal{C}^+$.

- Theorem (Thomason): if $\pi_n X = 0$ for $n < 0$ then $X \simeq K(\mathcal{C})$ for some $\mathcal{C}$.

- $K(\{\text{finite sets \& bijections}\}) = S^0$

- $K(\{\text{finite } G\text{-sets}\}) = \Sigma^\infty B\{\text{transitive } G\text{-sets}\}_+ = \bigvee_{(H)} BN_G H_+ = (S^0_G)^G$
  $K(\{\text{finite free } G\text{-sets}\}) = \Sigma^\infty BG_+$.

- $K(\{\text{fin. gen. proj. } R\text{-modules}\}) =: K(R)$. This has been studied extensively especially when R is a number field. (eg Voevodsky's Fields medal).

- If $\mathcal{C}$ is the 1+1-dimensional cobordism category (as in TQFT) then $B\mathcal{C}^+$ is related to mapping class groups & $K(\mathcal{C})$ is essentially $(\mathbb{C}P^\infty)^{-T}$ (a theorem of Madsen & Weiss, improving a longstanding conjecture of Mumford).

Goodwillie calculus

- For any reasonable functor F from spaces to spaces, Goodwillie has defined "derivatives" $D_n F$ and "Taylor approximations" $P_n F$.

- $D_n F$ is a spectrum with an action of the symmetric group Σ_n. Where the formulae of ordinary calculus have a denominator of $n!$, Goodwillie instead takes a quotient by an action of Σ_n.

- The identity functor is **not** linear. The first derivative $D_1 \mathrm{Id}$ is just S^0. The higher derivatives are related to partition complexes, the Bruhat-Tits building, Steinberg modules, symmetric powers of S^0, and K-theory of finite sets with multiplicities.

- There are variants of Goodwillie calculus for functors from other categories $\mathcal{C}$ to spaces.

    - One can study the space $\mathrm{Emb}(M, N)$ (embeddings of M in N, where M and N are manifolds) by taking $\mathcal{C} = \{$open subsets of $M\}$ and $F(U) = \mathrm{Emb}(U, N)$.

    - Related ideas are used in the proof of the Madsen-Weiss theorem.

    - There is also a good theory for functors from vector spaces to topological spaces.

- There are close relationships with surgery, pseudoisotopy theory, Waldhausen's K-theory of spaces, and so on.

Representability and exactness

- A cohomology theory is a contravariant functor $X \mapsto E^*X$ such that (i) $E^*\Sigma X = E^{*-1}X$ (ii) $E^*\bigvee_i X_i = \prod_i E^*X_i$
  (iii) $E^*X \leftarrow E^*Y \leftarrow E^*Cf$ is exact for any $f : X \to Y$.

- A homology theory is a covariant functor $X \mapsto E_*X$ with analogous properties.

- Theorem (Brown, Adams):
  - For any cohomology theory E^* there is a representing spectrum E such that $E^nX = [X, \Sigma^n E]$
  - For any homology theory E_* there is a representing spectrum E such that $E_nX = \pi_n(E \wedge X)$

- Examples: HA, MO, MU, $K(p,n)$, KU, ...

- The functor $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$ is represented by a spectrum I called the Brown-Comenetz dual of S^0. It is a fertile source of counterexamples.

- If M_* is a flat module over MU_* then $X \mapsto M_* \otimes_{MU_*} MU_*X$
  is a homology theory, represented by a spectrum M say. Flatness can be relaxed to Landweber exactness, a formal group theory condition that is easily checked in practice. Many concrete examples

- There is a spectrum $S_{(p)}$ such that $\pi_*(X \wedge S_{(p)}) = \pi_*(X)_{(p)}$. We write $X_{(p)} = X \wedge S_{(p)}$, so $\pi_*(X_{(p)}) = \pi_*(X)_{(p)}$.

Bousfield localization

- If we are using a homology theory E_* as our main tool, it makes sense to consider $\mathrm{ann}(E) = \{X \mid E_* X = 0\}$ and factor through the quotient category $\mathcal{B}/\mathrm{ann}(E)$.

- Put $\mathcal{B}_E = \{X \in \mathcal{B} \mid [W, X] = 0 \text{ for all } W \in \mathrm{ann}(E)\} \subseteq \mathcal{B}$

  There is an equivalence $L_E : \mathcal{B}/\mathrm{ann}(E) \xrightarrow{\sim} \mathcal{B}_E$ called Bousfield localization.

- Put $\mathcal{K}(p,n) = \mathcal{B}_{K(p,n)}$. These categories have much better finiteness properties than $\mathcal{B}$, reminiscent of modules for a complete Noetherian local ring. There is a close relationship with the Morava stabiliser group.

- Put $E(p,n) = K(p,0) \vee \dots \vee K(p,n)$ and $\mathcal{L}(p,n) = \mathcal{B}_{E(p,n)}$.

  The localization functor $L_{(p,n)} : \mathcal{B} \to \mathcal{L}(p,n)$ has the unusual property that $L_{(p,n)} X = X \wedge L_{(p,n)} S^0$ for all X

- The chromatic convergence theorem says that $X_{(p)} = \varprojlim_n L_{(p,n)} X$

  The difference between $L_{(p,n)} X$ and $L_{(p,n-1)} X$ is determined by the image of X in $\mathcal{K}(p,n)$. This gives an effective method for recovering information in $\mathcal{B}$ from the categories $\mathcal{K}(p,n)$.

- This chromatic programme is based on conjectures of Ravenel in the 1970's, now mostly proved by various subsets of Hopkins, Devinatz, Smith, Ravenel & Mitchell.

Rigidification

- So far, we have considered spectra only as homotopical objects, so $\mathcal{B}$ is analogous to the category of spaces and homotopy classes of maps.

- There is an underlying geometric category $\mathcal{M}$ from which $\mathcal{B}$ can be obtained by inverting weak equivalences (but not by the more naive procedure of passing to homotopy classes of maps.)

- This allows us to work with spectra with group actions, continuous families of spectra, diagrams of spectra and so on. There is a rich theory for group actions.

- We can also consider **strictly commutative ring spectra** ie spectra $E \in \mathcal{M}$ with a commutative and associative product $\mu : E \wedge E \to E$. Examples include H, MU and KU but not $K(p,n)$. There is a good theory of modules over these rings, and their K-theory and Galois theory.

- The theory evolved historically in a much more complex way than one would expect. For a long time there was no good candidate for $\mathcal{M}$; the eventual construction of $\mathcal{M}$ and analysis of its properties contained a number of surprises.